

Wavelet Transform on the Circle and the Real Line: a Unified Group-Theoretical Treatment *

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Abstract

We present a unified group-theoretical derivation of the Continuous Wavelet Transform (CWT) on the circle \mathbb{S}^1 and the real line \mathbb{R} , following the general formalism of Coherent States (CS) associated to unitary square integrable (modulo a subgroup, possibly) representations of the group $SL(2, \mathbb{R})$. A general procedure for obtaining unitary representations of a group G of affine transformations on a space of signals $L^2(X, dx)$ is described, relating carrier spaces X to (first or higher-order) “polarization subalgebras” \mathcal{P}_X . We also provide explicit admissibility and continuous frame conditions for wavelets on \mathbb{S}^1 and discuss the Euclidean limit in terms of group contraction.

1 Introduction

It is not straightforward to define a proper dilation operator on the circle, and therefore the wavelets on the circle cannot be attained by an irreducible representation of the affine group in a straightforward way. Holschneider obtained in Ref. [1] wavelets on the circle from the standard ones $\gamma_{b,a}(x) = a^{-1}\gamma(\frac{x-b}{a})$ by means of periodization:

$$\gamma_{\theta,a}(x) = \sum_{n \in \mathbb{Z}} \frac{1}{a} \gamma\left(\frac{x - \theta + 2\pi n}{a}\right), \quad \theta \in \mathbb{S}^1, a \in \mathbb{R}^+, \quad (1)$$

for functions γ which decay sufficiently fast at infinity.

In Ref. [2], the same author proposed a CWT on the sphere \mathbb{S}^2 satisfying some natural requirements, but with a number of *ad hoc* assumptions (in particular, for dilations). Later, in Ref. [3], this proposal was derived, and the assumptions proved, in a group-theoretical setting, by means of an appropriate unitary representation of the Lorentz group $SO_0(3, 1)$, using the coherent state machinery (see [4] for a review). This construction was extended afterwards to

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higher dimensional spheres \mathbb{S}^{n-1} in [5] by using the Lorentz group in $n+1$ dimensions $SO_0(n, 1)$. However, the singular case of $n = 2$ (corresponding to the circle \mathbb{S}^1) does not fit in with the general case.

Here we identify the group of affine transformations of \mathbb{S}^1 and \mathbb{R} , namely $SL(2, \mathbb{R})$, and derive the CWT on both spaces in a unified manner, based on the construction of general CS associated to square integrable representations (modulo a subgroup [4]) of $SL(2, \mathbb{R})$.

To that purpose, we outline a powerful technique (explained in Sec. 3) for obtaining representations of Lie groups, which is a mixture of Mackey's induction procedure and Kostant-Kirillov coadjoint orbits method. The advantage of following this general procedure is that it naturally reproduces a number of *ad hoc* assumptions found in the literature (i.e.: concerning semi-invariance and multipliers, definition of dilations, Euclidean limits, etc) to define a genuine CWT on a manifold X . This group-approach to the CWT also unifies representations in different carrier spaces X , associating them to different "polarization subalgebras" \mathcal{P}_X (see later on Sec. 3 for a formal definition), and providing unitary isomorphisms or "polarization-changing operators" (e.g.: Fourier and Laplace Transforms) which relate them. We shall explain this technique with the help of the affine group in one dimension. This simple example will serve us as a preamble to tackle the more involved case of $SL(2, \mathbb{R})$, where affine and "circular" wavelets coexist as different representation spaces associated to non-equivalent polarization subalgebras.

The organization of the paper is as follows. In Section 2 we briefly sketch the construction of coherent states associated to group representations, following [4]. In Section 3 we describe a general procedure for obtaining representations of a group G of affine transformations on a space of signals $L^2(X, dx)$, by relating carrier spaces X to "polarization subalgebras" $\mathcal{P}_X \subset \mathcal{G}$, the Lie algebra of G . Inside this scheme, Fourier and Laplace transforms will appear as particular cases of what we define in general by: "polarization-changing operators". As already said, we exemplify this abstract construction with the simple case of the affine group in Section 4. In Section 5 we derive discrete and continuous series representations of $SL(2, \mathbb{R})$, relating them to wavelets on the real line and the circle, respectively. Associated to two different polarization subalgebras, $\mathcal{P}_{\mathbb{R}^+}$ and $\mathcal{P}_{\mathbb{C}^+}$, we realize discrete series (affine wavelets) on the half line \mathbb{R}^+ (Section 5.1) and on the right half complex plane \mathbb{C}^+ (Section 5.2) and provide the corresponding polarization-changing operator (which turns out to be the Laplace transform) in terms of the generating function of the generalized Laguerre polynomials (the kernel). A third (non-equivalent) polarization subalgebra $\mathcal{P}_{\mathbb{S}^1}$ of $sl(2, \mathbb{R})$ leads to a realization of continuous series representations on the circumference (Section 5.4); here we derive the CWT on \mathbb{S}^1 , using the coherent state machinery explained in Sec. 2, from an appropriate unitary representation of $SL(2, \mathbb{R})$ obtained by following the general procedure designed in Sec. 3. In particular, we provide admissibility conditions and prove that the corresponding family of coherent states constitutes a continuous frame, following the strategy of [3] for \mathbb{S}^2 . We also study the Euclidean limit and prove that the continuous series representation of $SL(2, \mathbb{R})$ on \mathbb{S}^1 contract to the usual wavelet representation of the affine group on \mathbb{R} in the limit $R \rightarrow \infty$ (large radius).

2 The CWT on a Manifold

The usual CWT on the real line $X = \mathbb{R}$ is derived from the natural unitary representation of the affine group $G = SIM(1)$ in the space of finite energy signals $L^2(\mathbb{R}, dx)$. The same scheme applies to the CWT on a general manifold X , subject to the transitive action, $x \rightarrow gx, g \in G, x \in X$,

of some group of transformations G which contains dilations. If the measure dx is G -invariant (i.e. $d(gx) = dx$), then the natural left action of G on $L^2(X, dx)$ given by:

$$[U^L(g)\psi](x) = \psi(g^{-1}x), \quad g \in G, \psi \in L^2(X, dx), \quad (2)$$

defines a unitary representation, that is:

$$\langle U^L(g)\psi | U^L(g)\phi \rangle = \langle \psi | \phi \rangle \equiv \int_X \bar{\psi}(x)\phi(x)dx. \quad (3)$$

When dx is not strictly invariant (i.e. $d(gx) = \lambda(g, x)dx$), we have to introduce a *multiplier*

$$[U^L(g)\psi](x) = \lambda(g, x)^{1/2}\psi(g^{-1}x), \quad g \in G, \psi \in L^2(X, dx), \quad (4)$$

in order to keep unitarity. The fact that $U^L(g_2)U^L(g_1) = U^L(g_2g_1)$ (i.e. U^L is a representation of G) implies cohomology conditions for multipliers, that is:

$$\lambda(g_2g_1, x) = \lambda(g_2, x)\lambda(g_1, g_2^{-1}x). \quad (5)$$

In this case, multipliers are associated with the Radon-Nikodym derivative of the quasi-invariant measure. However, we shall show how multipliers naturally emerge from our scheme in a non-ad-hoc manner (see next Section).

Let us consider the space $L^2(G, d^Lg)$ of square-integrable complex functions Ψ on G , where $d^Lg = d^L(g'g)$, $\forall g' \in G$, stands for the left-invariant Haar measure, which defines the scalar product:

$$(\Psi | \Phi) = \int_G \bar{\Psi}(g)\Phi(g)d^Lg. \quad (6)$$

A non-zero function $\gamma \in L^2(X, dx)$ is called *admissible* (also “wavelet analyzing function”) if $\Gamma(g) \equiv \langle U^L(g)\gamma | \gamma \rangle \in L^2(G, d^Lg)$, that is, if

$$C_\gamma = \int_G \bar{\Gamma}(g)\Gamma(g)d^Lg = \int_G |\langle U^L(g)\gamma | \gamma \rangle|^2 d^Lg < \infty. \quad (7)$$

Let us assume that the representation U^L is *irreducible*, and that there exists a function γ admissible, then a system of coherent states (CS) on X associated to (or indexed by) G are defined as the functions in the orbit of γ under G :

$$\gamma_g = U^L(g)\gamma, \quad g \in G. \quad (8)$$

When the representation U^L is not square integrable, it is not possible to find admissible functions γ since $\Gamma(g)$ is not square integrable. We can still proceed by restricting ourselves to a suitable homogeneous space $Q = G/H$, for some closed subgroup H . Then, the non-zero function γ is said to be admissible mod(H, σ) (with $\sigma : Q \rightarrow G$ a Borel section), and the representation U^L square integrable mod(H, σ), if the following condition holds:

$$\int_Q |\langle U^L(\sigma(q))\gamma | \psi \rangle|^2 d^Lq < \infty, \quad \forall \psi \in L^2(X, dx), \quad (9)$$

where d^Lq is a measure on Q “projected” from the left-invariant measure d^Lg on the whole G . The coherent states indexed by Q are defined as $\gamma_{\sigma(q)} = U^L(\sigma(q))\gamma$, $q \in Q$, and they form an overcomplete set in $L^2(X, dx)$.

The condition (9) could also be written as an expectation value

$$0 < \int_Q |\langle U^L(\sigma(q))\gamma|\psi\rangle|^2 d^L q = \langle \psi|A_\sigma|\psi\rangle < \infty, \quad \forall \psi \in L^2(X, dx), \quad (10)$$

where $A_\sigma = \int_Q |\gamma_{\sigma(q)}\rangle\langle\gamma_{\sigma(q)}| d^L q$ is a positive, bounded, invertible operator. If the operator A_σ^{-1} is also bounded, then the set $S_\sigma = \{\gamma_{\sigma(q)}, q \in Q\}$ is called a *frame*, and a *tight frame* if A_σ is a positive multiple of the identity, $A_\sigma = \lambda \mathbb{I}, \lambda > 0$.

To avoid domain problems in the following, let us assume that γ generates a frame (i.e. that A_σ^{-1} is bounded). The *CS map* or *wavelet transform* is defined as the linear map

$$\begin{aligned} T_\gamma : L^2(X, dx) &\longrightarrow L^2(Q, d^L q) \\ \psi &\longmapsto \Psi_\gamma(q) = [T_\gamma \psi](q) = \frac{\langle \gamma_{\sigma(q)}|\psi\rangle}{\sqrt{c_\gamma}}, \end{aligned} \quad (11)$$

where c_γ is the squared norm of $\Gamma(q) \equiv \langle U^L(\sigma(q))\gamma|\gamma\rangle \in L^2(Q, d^L q)$. Its range $L^2_\gamma(Q, d^L q) \equiv T_\gamma(L^2(X, dx))$ is complete with respect to the scalar product $(\Phi|\Psi)_\gamma \equiv (\Phi|T_\gamma A_\sigma^{-1} T_\gamma^{-1} \Psi)_Q$ and T_γ is unitary from $L^2(X, dx)$ onto $L^2_\gamma(Q, d^L q)$. Thus, the inverse map T_γ^{-1} yields the *reconstruction formula*:

$$\psi = T_\gamma^{-1} \Psi_\gamma = \int_Q \Psi_\gamma(q) A_\sigma^{-1} \gamma_{\sigma(q)} d^L q, \quad \Psi_\gamma \in L^2_\gamma(Q, d^L q), \quad (12)$$

which expands the signal ψ in terms of CS $A_\sigma^{-1} \gamma_{\sigma(q)}$ with wavelet coefficients $\Psi_\gamma(q) = [T_\gamma \psi](q)$. These formulas acquire a simpler form when A_σ is a multiple of the identity.

3 Obtaining the representations of G on $L^2(X, dx)$

In this section we shall explain the procedure to obtain the irreducible representations of a group G on a space $L^2(X, dx)$, where $X = G/H$ is an homogeneous space under G , from the left action of G on complex (Borel) functions Ψ over G , $\mathcal{F}(G)$:

$$[U^L(g)\Psi](g') = \Psi(g^{-1}g'), \quad \Psi \in \mathcal{F}(G) \quad (13)$$

This procedure is a mixture of Mackey's induction technique and Kostant-Kirillov coadjoint orbit method, with some new ingredients such as higher-order polarizations. It is known as Group Approach to Quantization (GAQ) [6, 7], although in this paper we shall use non-horizontal polarizations instead of pseudo-extensions [8].

The idea is to consider the representation induced by a certain one-dimensional representation D_α of a subgroup P of G , restricting the representation (13) to the subspace $\mathcal{H}_\alpha \subset \mathcal{F}(G)$ of (Borel) functions on G satisfying:

$$\Psi(gh) = D_\alpha(h)\Psi(g), \quad \forall g \in G, \forall h \in P \quad (14)$$

We shall choose P appropriately to obtain an irreducible representation of G (see below about irreducibility).

For simplicity we shall consider connected and simply connected Lie groups. In the case the group is not simply connected, we study the representations of the universal covering group and then determine which representations of the universal covering group are also representations of the original group.

We shall also suppose that P is connected and simply connected. Under this conditions, we can work at the infinitesimal level and use left invariant vector fields to implement (14) and reduce in this way the representation (13), which is highly reducible. Left-invariant vector fields, once a set of local coordinates $\{g^i\}_{i=1}^{\dim G}$ is chosen, are easily computed as

$$X_j^L(g) \equiv X_{g^j}^L(g) = \sum_{k=1}^{\dim G} (L_g^T(e))_j^k \frac{\partial}{\partial g^k}, \quad (L_g^T(e))_j^k = \left. \frac{\partial (gg')^k}{\partial g'^j} \right|_{g'=e}, \quad j, k = 1, \dots, \dim G.$$

Here $L_g^T(e)$ represents the tangent at the identity element of the left translation on the group $L_g(g') = gg'$, and $X_j^L(g)$ is the left-invariant vector field verifying $X_j^L(e) = \frac{\partial}{\partial g^j}$ (see, for instance, [9]). These vector fields are left-invariant thanks to the chain rule, and they turn out to be the infinitesimal generators of the right action of the group on $\mathcal{F}(G)$, $[U^R(g')\Psi](g) = \Psi(gg')$. Where there is no confusion, we shall omit the dependence on the group element, and write X_j^L for $X_j^L(g)$.

The set of left-invariant vector fields $\{X_j^L\}_{j=1}^{\dim G}$ is a basis of $T_g G$ closing a Lie algebra isomorphic to the Lie algebra \mathcal{G} of G , and therefore they constitute a realization of it acting on $\mathcal{F}(G)$.

Thus, if \mathcal{P} is the Lie algebra of P , realized in terms of left-invariant vector fields, we impose a condition of the form

$$X^L \Psi(g) = \alpha(X^L) \Psi(g), \quad \forall X^L \in \mathcal{P}, \quad (15)$$

where now $\Psi \in C^1(G)$ and α constitutes a one-dimensional representation (character) of the subalgebra \mathcal{P} , which is the infinitesimal character associated to D_α . Since the representation is one-dimensional, the character α vanishes on the derived subalgebra of \mathcal{P} , $\alpha([X^L, Y^L]) = 0, \forall X^L, Y^L \in \mathcal{P}$. This means that the character α can be non-trivial only on the quotient of \mathcal{P} by its derived algebra, $\mathcal{A} \equiv \mathcal{P}/[\mathcal{P}, \mathcal{P}]$ which is an Abelian algebra. In particular if \mathcal{P} is semisimple, the character α is trivial.

The subalgebra \mathcal{P} will be called a polarization, more precisely, a first-order polarization[†].

Definition 3.1. A **first-order polarization** is a proper subalgebra \mathcal{P} of the Lie algebra \mathcal{G} of G , realized in terms of left-invariant vector fields.

Usually we shall be interested in first-order polarizations such that the representation obtained after the polarization conditions are imposed, see eq. (22), is not one-dimensional.

The advantage of using polarization equations is two-fold. On the one hand we have at our disposal the powerful machinery of partial differential equations, Frobenius theorem, etc., and, on the other hand, we can generalize them to account for higher-order differential operators. Thus, we can define:

Definition 3.2. A **higher-order polarization** is a proper subalgebra \mathcal{P}^{HO} of the (left) universal enveloping algebra \mathcal{UG} of \mathcal{G} .

Higher-order polarization conditions are imposed in a similar way to (15):

$$X^{HO} \Psi(g) = \alpha(X^{HO}) \Psi(g), \quad \forall X^{HO} \in \mathcal{P}^{HO} \quad (16)$$

[†]Originally, the definitions of first-order and higher-order polarizations were associated with a central extension \tilde{G} of G (which defines a notion of horizontality in the Lie algebra), since they were introduced in the framework of Geometric Quantization [10, 11, 12, 13]. Here we adapt the definition to an arbitrary group, and this results in a larger freedom in the choice of polarizations.

where $\Psi \in C^k(G)$, k being the maximum degree of the differential operators in \mathcal{P}^{HO} (and can actually be infinity), and α is a 1-dimensional representation of \mathcal{P}^{HO} . Here we also shall be interested in higher-order polarization leading to non one-dimensional representations.

An important fact about polarizations is that they are chosen in such a way that they are compatible with the action of the group in (13). In fact, since left and right actions always commute in a group, the conditions (15) (resp. (16)) imposed by the polarization equations (left-invariant vector fields which generate the right finite group action) are preserved by the left action defined by (13). This means that if $\mathcal{H}_\alpha \subset C^1(G)$ (resp. $C^k(G)$) is the space of solutions of (15) (resp. (16)), then $U^L \mathcal{H}_\alpha \subset \mathcal{H}_\alpha$. This is one of the main features of GAQ, in contrast to other approaches like Geometric Quantization where the existence of polarizations compatible with a given group action is not always guaranteed.

Conditions like (16) imposed by higher-order polarizations have no counterpart at the level of finite group transformations. This is one of the main advantages of using polarizations instead of equations like (14). Except for certain cases, denoted anomalous [14], where specific representations require the use of higher-order polarizations, almost all (unitary) irreducible representations of G can be obtained by means of first-order polarizations. But even in these cases, higher-order polarizations are useful for obtaining realizations in certain carrier spaces X which are not homogeneous spaces, that is, that are not of the form $X = G/H$ for any H subgroup of G .

3.1 First-order polarizations

Let us study in detail the case of first-order polarizations, in particular the structure of the space \mathcal{H}_α of solutions of (15) (or its finite group transformations counterpart (14)). This is a system of linear homogeneous first-order partial differential equations, and the form of its solutions can be described as follows.

Proposition 3.3. *The solutions of the equations (15) can be factorized as $\Psi(g) = W_\alpha(g)\Psi_0(g)$, where $\Psi_0(g)$ is the general solution of the $\alpha = 0$ system (trivial representation of \mathcal{P}):*

$$X^L \Psi_0(g) = 0, \quad \forall X^L \in \mathcal{P}, \quad (17)$$

and $W_\alpha(g)$ is a particular solution of the $\alpha \neq 0$ system.

Proof: Since vector fields are derivations, they satisfy Leibnitz's rule, and therefore $W_\alpha(g)\Psi_0(g)$ is a solution of (15) provided they satisfy the hypothesis of the proposition. To prove that any solution can be written in this way, it suffices to show that an everywhere non vanishing solution $W_\alpha(g)$ exist, and then $W_\alpha^{-1}(g)\Psi(g)$ satisfies the $\alpha = 0$ system.

According to Levi-Malcev theorem (see [15]), \mathcal{P} can be decomposed as the semidirect product of a solvable algebra \mathcal{R} (the radical) by a semisimple algebra \mathcal{S} : $\mathcal{P} = \mathcal{S} \ltimes \mathcal{R}$. The 1-dim representation α is trivial on \mathcal{S} , and therefore is determined by the radical \mathcal{R} . According to Lie's theorem (see [16]), a set of operators closing a solvable algebra can always be simultaneously diagonalized:

$$X^L W_\alpha(g) = \alpha(X^L) W_\alpha(g), \quad \forall X^L \in \mathcal{R}. \quad (18)$$

where α is a 1-dim representation of \mathcal{R} . Since α is trivial on the ideal $[\mathcal{R}, \mathcal{R}]$, the solutions of (18) are determined by the solutions for the Abelian algebra $\mathcal{A} = \mathcal{R}/[\mathcal{R}, \mathcal{R}]$ which turns to equal $\mathcal{P}/[\mathcal{P}, \mathcal{P}]$. Let A be the quotient group $P/[P, P]$ where $[P, P]$ denotes the commutator group of

P (note that since we have supposed P simply connected, $[P, P]$ is a closed subgroup, see [9]). Let $W_\alpha^A(a)$ be a particular solution of the equations

$$X^L W_\alpha^A = \bar{\alpha}(X^L) W_\alpha^A, \quad \forall X^L \in \mathcal{A}, \quad (19)$$

which again exists by Lie's theorem, and where $\bar{\alpha}$ denotes the character of \mathcal{A} which extends to the character α of \mathcal{R} . Using Frobenius theorem, we can choose local coordinates $\{a^i\}$ in A in such a way that $X_i^L = \frac{\partial}{\partial a^i}$, $\forall X_i^L \in \mathcal{A}$, and therefore $W_\alpha^A(a) = e^{\sum_{i=1}^{\dim A} \bar{\alpha}_i a^i}$, where we denote $\bar{\alpha}_i = \bar{\alpha}(X_i^L)$, $X_i^L \in \mathcal{A}$. In particular $W_\alpha^A(a)$ can be chosen to be non-zero everywhere. The solution $W_\alpha^A(a)$ on A can be extended to a particular solution $W_\alpha(g)$ of (18), $W_\alpha(g) = e^{\sum_{i=1}^{\dim A} \alpha_i a^i(g)}$, which satisfy the required properties. ■

Let us study in detail the properties of the solutions Ψ_0 of (17). The finite group transformations version of this equation is written in terms of the right action U^R of G on $\mathcal{F}(G)$ as:

$$[U^R(h)\Psi_0](g) \equiv \Psi_0(gh) = \Psi_0(g), \quad \forall h \in P, \quad (20)$$

although in this case the differentiability requirement can be dropped, so $\Psi_0 \in \mathcal{F}(G)$. Let $\mathcal{H}_0 \subset \mathcal{F}(G)$ be the space of solutions of the last equation. If we denote by $X = G/P$, then we have the following result:

Proposition 3.4. *There is an isomorphism between the space of solutions \mathcal{H}_0 of (20) and the space of complex (Borel) functions $\mathcal{F}(X)$ on X .*

Proof: Let $\pi : G \rightarrow X = G/P$ be the canonical projection. Then, for any $\phi \in \mathcal{F}(X)$ we can define a function $\Psi_0 \in \mathcal{F}(G)$ by $\Psi_0(g) = \phi(\pi(g))$. This function verifies (20) since $\Psi_0(gh) = \phi(\pi(gh)) = \phi(\pi(g)) = \Psi_0(g)$.

On the other hand, if $\Psi_0 \in \mathcal{H}_0$ is a solution of (20) and $s : X \rightarrow G$ is a Borel section, we can define a function $\phi \in \mathcal{F}(X)$ by $\phi(x) = \Psi_0(s(x))$, $\forall x \in X$. The function ϕ is independent of the choice of Borel section, since if s' is another Borel section, then $s'(x) = s(x)h(x)$, with $h(x) \in P$, $\forall x \in X$ (a gauge transformation). Then, using (20) we have that $\phi'(x) = \Psi_0(s'(x)) = \Psi_0(s(x)h(x)) = \Psi_0(s(x)) = \phi(x)$. Therefore, ϕ is unique. ■

This result states that the space \mathcal{H}_α of solutions of (15) (or rather the corresponding subspace of $\mathcal{F}(G)$ of solutions of the finite group transformations counterpart (14)) is isomorphic to $\mathcal{F}(X)$, since the function $W_\alpha(g)$ is fixed. We can translate the action $[\mathcal{U}_\alpha(g)\Psi](g')$ of $g \in G$ on $\Psi = W_\alpha\Psi_0 \in \mathcal{F}(G)$ onto $\Psi_0 \in \mathcal{F}(X)$ in a suitable manner. First, let us define on \mathcal{H}_0 a modified action:

$$[\mathcal{U}_\alpha(g)\Psi_0](g') \equiv W_\alpha^{-1}(g')[U^L(g)(W_\alpha\Psi_0)](g') = W_\alpha^{-1}(g')W_\alpha(g^{-1}g')\Psi_0(g^{-1}g') \quad (21)$$

where $\Psi_0 \in \mathcal{H}_0$. Now the function $\lambda_\alpha(g, g') \equiv W_\alpha^{-1}(g')W_\alpha(g^{-1}g')$ is really a function on $G \times X$, since $\lambda_\alpha(g, g'h) = W_\alpha^{-1}(g'h)W_\alpha(g^{-1}g'h) = D_\alpha(h)^{-1}W_\alpha^{-1}(g')D_\alpha(h)W_\alpha(g^{-1}g') = W_\alpha^{-1}(g')W_\alpha(g^{-1}g') = \lambda_\alpha(g, g')$, provided that W_α is a particular solution of (14).

The function $\lambda_\alpha(g, x)$ is a multiplier and satisfies 1-cocycle properties, see (5). This means that $\mathcal{U}_\alpha(g)$ constitutes a representation of G on \mathcal{H}_0 .

The representation on $\mathcal{F}(X)$ is now defined, using a Borel section $s : X \rightarrow G$, as:

$$\begin{aligned} \mathcal{U}_\alpha(g)\phi(x) &\equiv W_\alpha^{-1}(s(x))U^L(g)W_\alpha(s(x))\Psi_0(s(x)) \\ &= W_\alpha^{-1}(s(x))W_\alpha(g^{-1}s(x))\Psi_0(g^{-1}s(x)) = \lambda_\alpha(g, s(x))\phi(\pi(g^{-1}s(x))) \\ &= \lambda_\alpha(g, x)\phi(g^{-1}x) \end{aligned} \quad (22)$$

where $\Psi_0(g) \equiv \phi(\pi(g))$ and gx is the natural action of G on the coset space $X = G/P$. This representation does not depend on the choice of “representative” Borel section s , since λ_α is a function on $G \times X$.

The infinitesimal version of this representations can be easily computed. Since right and left-invariant vector fields commute, the form of the solutions will be preserved by the action of right-invariant vector fields, i.e. $X^R W_\alpha(g) \Psi_0(g) = W_\alpha(g) \Psi'_0(g)$. Thus, we can define:

$$\mathcal{X}_j \phi(x) \equiv W_\alpha^{-1}(s(x))(iX_j^R)W_\alpha(s(x))\Psi_0(s(x)) \quad j = 1, \dots, \dim(G), \quad (23)$$

where the imaginary constant is introduced generally to obtain hermitian operators, and where $\Psi_0(g) \equiv \phi(\pi(g))$.

To discuss the unitarity of the obtained representations, we need to introduce a scalar product, that is, an invariant or quasi-invariant measure on X . The existence of invariant measures on homogeneous spaces is not always guaranteed, but there always exist quasi-invariant measures [17]. Given a quasi-invariant measure $d\mu(x)$ on X , we have to introduce the corresponding Radon-Nikodym derivative $\lambda(g, x) \equiv \frac{d\mu(gx)}{d\mu(x)}$.

In our approach we are able to construct quasi-invariant measures and the corresponding Radon-Nikodym derivative in a easy way (see [18] for details). The invariant or quasi-invariant measure is given by:

$$d\mu = i_{X_p^L} i_{X_{p-1}^L} \cdots i_{X_1^L} d^L g, \quad X_1^L, X_2^L, \dots, X_p^L \in \mathcal{P} \quad (24)$$

where $\{X_1^L, X_2^L, \dots, X_p^L\}$, $p = \dim P$, is a basis of \mathcal{P} and $i_X \omega$ denotes interior product (or contraction) of the vector field X and the differentiable form ω . As proved in [18], $d\mu$ is an invariant measure on $X = G/P$ or can be converted into a quasi-invariant one by multiplying it by an appropriate function ρ which satisfies an equation of the type (14). In fact, the function ρ is of the form $\rho(g) = |W_\alpha(g)|^2$ for a particular choice of the 1-dim representation D_α . The corresponding Radon-Nikodym derivative is given by:

$$\lambda(g, x) = \frac{\rho(gs(x))}{\rho(s(x))} \quad (25)$$

where $s : X \rightarrow G$ is any Borel section. The advantage of our procedure is that the function ρ is built in the solutions of the equation (14), in such a way that the quadratic form $\bar{\Psi}\Psi'$ appearing in the scalar product leads to $|W_\alpha(g)|^2 \bar{\Psi}_0(g) \Psi'_0(g)$ which can be properly integrated on X with respect to the measure $d\mu(x)$.

In this way, choosing appropriately the 1-dim representation D_α , the representation \mathcal{U}_α can be made unitary acting on $L^2(X, |W_\alpha(g)|^2 d\mu)$.

Note that if X does not admit an invariant measure, then the representation D_α must be non-unitary. Otherwise the function $\rho(g) = |W_\alpha(g)|^2$ belongs to \mathcal{H}_0 and the Radon-Nikodym derivative is trivial or it is a 1-coboundary (see below).

Choosing different particular solutions W_α and W'_α of (18) (or (14)) leads to (unitarily) equivalent representations. We only need to take into account that $W'_\alpha(g) = W_\alpha(g)\rho(g)$, with $\rho \in \mathcal{H}_0$. Then, if $s : X \rightarrow G$ is a Borel section:

$$\lambda'_\alpha(g, s(x)) = W_\alpha'^{-1}(s(x))W'_\alpha(g^{-1}s(x)) = W_\alpha^{-1}(s(x))W_\alpha(g^{-1}s(x))\frac{\rho(g^{-1}s(x))}{\rho(s(x))} \quad (26)$$

Now the function $\lambda(g, x) = \frac{\rho(g^{-1}s(x))}{\rho(s(x))}$ is a trivial 1-coycle, i.e., a 1-coboundary generated by the 0-cocycle $\rho(s(x))$. This means that the multipliers λ_α and λ'_α are equivalent (see [19]) and therefore the defined representations \mathcal{U}_α and \mathcal{U}_α' are (unitarily) equivalent.

Let us discuss on the irreducibility of the representations obtained by first order polarizations.

Definition 3.5. *A unitary representation of G is said to be **quasi-irreducible** if the set of operators commuting with the representation, up to multiplicative constants, is discrete.*

A unitary quasi-irreducible representation decomposes into a discrete sum of irreducible representations.

Definition 3.6. *A first-order polarization \mathcal{P} is said to be **full** (or **regular**) if the representation obtained after the polarization conditions are imposed, eq. (22), is irreducible or quasi-irreducible.*

In the case of a full polarization leading to a quasi-irreducible representation, we shall need to resort to a non-connected subgroup $P \subset G$, or even to discrete operators external to the group to fully reduce the representation (see for instance Sec. 4.1, after eq. (48), and Sec. 5.4, after eq. (94)).

A non-full first-order polarization \mathcal{P} will lead to a reducible representation. The (quasi-) irreducibility will be accomplished by enlarging \mathcal{P} with elements either of the Lie Algebra, thus leading to a larger full first-order polarization, or of the enveloping algebra $\mathcal{U}(\mathcal{G})$, thus leading to a higher-order polarization.

Another problem is that the polarization subgroup P can be multiply connected, even though we have supposed that G is simply connected (think, for instance, in $U(1) \subset SU(2)$). In this case, not all infinitesimal characters α of \mathcal{P} are exponentiable to a character D_α of P , and some restrictions appear (like half-integer values of spin for $SU(2)$).

In certain circumstances, it is convenient to resort to complex polarizations $\mathcal{P} \subset \mathcal{G}^c$ in order to obtain representations on carrier spaces with a Kähler structure (as, for instance, for the discrete series of representations for semisimple Lie groups). In this case, the use of polarization subalgebras avoids the need of complexifying the group, only the polarization is complex and the carrier space of the representation is $X = G/H$ where H is a subgroup whose Lie algebra is a real subalgebra of \mathcal{P} . The space X admits a Kähler structure, and in this case the function $W_\alpha(g)$ is related to the Kähler potential.

Finally, let us comment that we have chosen to impose polarizations with finite right group actions U^R (i.e. with left-invariant vector fields X^L) and group actions with finite left group actions U^L . Nevertheless, left and right can be interchanged at computational convenience, and the representations obtained are (unitarily) equivalent. Thus, we shall choose the more convenient option along the examples of this paper.

3.2 Higher-order polarizations

Although higher-order polarizations have been widely used in GAQ, there are not many conclusive results describing the general properties of the solution spaces and the unitarity of the representations (as the existence of invariant measures, multipliers, etc.). However we shall give an operative description of how they work (see for instance, [10, 11, 12, 13]).

The first difference with first-order polarizations lies in the fact that Leibnitz's rule does not apply for higher-order differential operators, therefore factorizations like the one used in the

previous subsection are not valid. That is, the general solution of (16) cannot be written as $\Psi(g) \neq W_\alpha(g)\Phi(g)$. This implies, in particular, that the space of solutions $\mathcal{H}_\alpha \subset C^k(G)$ is not isomorphic to $C^k(X)$, with $X = G/H$, for any subgroup H of G . That is, the carrier space of the representation will not be an homogeneous space.

In many examples studied, the general solution of (16) turns out to have the form:

$$\Psi(g) = \hat{W}_\alpha(g)\Phi(x) \quad (27)$$

where $\hat{W}_\alpha(g)$ is an infinite-order (pseudo-)differential operator acting on $\Phi \in C^\infty(X)$ and X is a non-homogeneous submanifold of G . With the appropriate coordinates, $\hat{W}_\alpha(g)$ can be written as:

$$\hat{W}_\alpha(g) = e^{\sum_i s^i(g)\hat{O}_i(x)}, \quad (28)$$

where i runs from $i = 1, \dots, \dim(\mathcal{P}^{HO})$ and $\hat{O}_i(x)$ denote a set of higher-order differential operators acting on $C^\infty(X)$ related to the higher-order polarization. An example of this will be given in Sec. 5.1, eqs. (67) and (68).

Once the solutions of the higher-order polarization have been found, the representation (13) can be reduced to the space X . In this case, the finite group action cannot be computed, since it is not transitive, but we can compute the infinitesimal action by means of right-invariant vector fields. Since right and left-invariant vector fields commute, the form (27) of the solutions will be preserved by the action of right-invariant vector fields, i.e. $X^R \hat{W}_\alpha(g)\Phi(x) = \hat{W}_\alpha(g)\Phi'(x)$. Thus, $\hat{W}_\alpha^{-1}(g)X^R \hat{W}_\alpha(g)\Phi(x)$ is a well-defined operator on $C^\infty(X)$, and we can define:

$$\mathcal{X}_j\Phi(x) \equiv \hat{W}_\alpha^{-1}(g)(iX_j^R)\hat{W}_\alpha(g)\Phi(x) \quad j = 1, \dots, \dim(G). \quad (29)$$

This defines a representation of G on $C^\infty(X)$ by means of pseudo-differential operators (see eq. (69) for the above-mentioned example). This representation can be converted to a unitary representation on $L^2(X, d\mu)$ if an appropriate measure $d\mu$ is chosen (see Sec. 5.1, after eq. (70)). However, there are no conclusive results concerning the existence of invariant or quasi-invariant measures for higher-order polarizations (even these notions should be properly defined for non-homogeneous spaces).

In some cases, the operators \mathcal{X}_i should be modified in order to obtain a unitary representation, in a process that can be seen as the generalization for higher-order polarizations of the Radon-Nikodym derivative (see [20]).

In most cases, a higher-order polarization is obtained by adding higher-order operators to a first order, non-full, polarization. In particular, Casimir operators are very useful for this purpose, since they are central in the enveloping algebra and therefore can be added to any polarization (first or higher-order).

As far as irreducibility is concerned, the same considerations as in the case of first-order polarizations apply.

In certain cases, (real) higher-order polarizations can be used instead of complex polarizations. In this case, we either obtain the realization in the same Kähler manifold (see for instance Sec. 5.2), or a representation on a non-homogeneous manifold is obtained, like in the case of the harmonic oscillator in configuration space, either non-relativistic or relativistic (see [21]).

3.2.1 Second-order polarizations

The case of second-order polarizations merits special attention, since some results from the theory of partial differential equations can be applied to this case. Also, all higher-order polar-

izations appearing in this work are of second order type.

Second-order partial differential equations can be classified into three types, according to the eigenvalues of the coefficient matrix of the second order terms. They can be elliptic, if all eigenvalues are positive (or all negative), hyperbolic, if one of the eigenvalues is negative and the rest are positive, and parabolic if some eigenvalues are zero.

For the case of $SL(2, \mathbb{R})$, the Casimir operator, see eq. (62), is a hyperbolic second-order differential operator, as can be checked by direct computation, or using the fact that the invariant Killing metric has signature $(+, +, -)$. Here, a second-order polarization consists of the Casimir and a left (or right) invariant vector field. We shall first solve the (first-order) equation associated with the vector field, and substitute the solution into the Casimir equation. This produces a second-order partial differential equation (with two independent variables) that can be hyperbolic, elliptic or parabolic.

The elliptic equation in this case is solved by factorization, using complex variables. The factorization of the second-order differential operator implies that the polarization equation is reduced to a first-order equation, and therefore the results of first-order polarization apply. An example of this can be found in Section 5.2. Also, in this section other possible solutions of the elliptic equation are considered.

The parabolic equation appears in Section 5.1, and in this case it has the form of a Schrödinger-like equation (Parabolic equations generally lead to Heat-type equations, but since we are considering unitary representations, the equations must imply a unitary evolution). This Schrödinger-like equation is solved by the method explained previously, leading to a representation that is genuinely of higher-order type.

The hyperbolic equation, which appears in Sec. 5.3, can be solved using a factorization method. In this case a representation with support in \mathbb{S}^1 is obtained, leading to the continuous series of representations, obtaining the same results as in Sec. 5.4 with a first-order polarization.

3.3 Polarization-changing operators

Not all polarizations lead to different, nonequivalent representations. In many cases, two different polarizations (even of different type, one first-order and the other higher-order) lead to equivalent representations which are however realized in different carrier spaces. In this case we would like to compute the unitary operator which relates both representations. In this context, we shall call them polarization-changing operators, and they can be constructed as follows.

Let us denote by \mathcal{P}_X the polarization (first or higher-order) leading to the representation \mathcal{U}_X (or series of representations) on the Hilbert space $L^2(X, dx)$, where dx is an appropriate measure on X that makes the representation unitary. Then, if the polarizations \mathcal{P}_X and \mathcal{P}_Y lead to equivalent representations \mathcal{U}_X and \mathcal{U}_Y , there exist a unitary isomorphism $V : L^2(X, dx) \rightarrow L^2(Y, dy)$ such that $\mathcal{U}_Y V = V \mathcal{U}_X$. The unitary operator V will be in general an integral operator characterized by a kernel

$$\tilde{\phi}(y) \equiv V\phi(x) = \int_X K(x, y)\phi(x)dx \quad (30)$$

For convenience, we shall use Dirac's bra-ket notation, and write $\phi(x) \equiv \langle x|\phi \rangle \in L^2(X, dx)$ and $\tilde{\phi}(y) \equiv \langle y|\phi \rangle \in L^2(Y, dy)$.[‡] In Dirac's notation bras and kets can be separated and have

[‡]In order to avoid confusion, we shall denote with $\phi(x)$ and $\tilde{\phi}(y)$ the realization of the same "state" $|\phi\rangle$ in different Hilbert spaces.

meaning by themselves: kets are vectors in an abstract Hilbert space and bras belong to its dual.

Completeness relations (resolutions of unity) and reproducing kernels [4] adopt a very intuitive form in Dirac's notation. For instance, we can formally write:

$$\tilde{\phi}(y) = \langle y|\phi \rangle = \int_X dx \langle y|x \rangle \langle x|\phi \rangle = \int_X dx \langle y|x \rangle \phi(x) \quad (31)$$

where the completeness relation $I = \int_X dx |x\rangle \langle x|$ has been introduced. The factor $\langle y|x \rangle$ in the integral is the kernel for the integral transform (31) which relates the representation on $L^2(X, dx)$ to the one in $L^2(Y, dy)$. To compute this integral kernel we can proceed as follows. Search for a basis of states $\{|n\rangle\}$ of the Hilbert space, where the index n is either discrete, continuous or of both types. This basis should be easily computed in both Hilbert spaces $L^2(X, dx)$ and $L^2(Y, dy)$. Then introduce the completeness relation $I = \int |n\rangle \langle n| d\mu(n)$:

$$\langle y|x \rangle = \int \langle y|n \rangle \langle n|x \rangle d\mu(n) \quad (32)$$

where $d\mu$ accounts for a discrete and/or continuous measure. For the basis $\{|n\rangle\}$ we generally use the eigenfunctions of some of the generators of the infinitesimal action of the group G on X , which are self-adjoint and therefore their spectrum are complete in the Hilbert space.

Examples of polarization-changing operators are given in [22, 21], where a relativistic counterpart of the Bargmann transform relating configuration space and Bargmann-Fock space for the harmonic oscillator is given, and in Sec. 4.2 and 5.2 where the Fourier and Laplace transform are derived. See also [23] for more examples.

4 Wavelets for the affine group

Let us consider the affine group in one dimension, $SIM(1)$, which will serve us as an introduction for studying the most interesting case of $SL(2, \mathbb{R})$ which will be considered in the next section.

The affine group $G = SIM(1) = \mathbb{R} \rtimes \mathbb{R}^+ = \{g = (b, a) / b \in \mathbb{R}, a \in \mathbb{R}^+\}$ of translations and dilations in one dimension is given by the following group law ($g'' = g'g$):

$$\begin{aligned} a'' &= a'a \\ b'' &= b + ab' \end{aligned} \quad (33)$$

This group law can be obtained from the upper triangular matrices with determinant one and positive diagonal, which admit the following Iwasawa KAN decomposition (with K trivial in this case):

$$g = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \sqrt{a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{a}} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{a} \end{pmatrix} \quad (34)$$

Left and right-invariant vector fields are immediately computed from the group law:

$$\begin{aligned} X_a^L &= a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} & X_a^R &= a \frac{\partial}{\partial a} \\ X_b^L &= \frac{\partial}{\partial b} & X_b^R &= a \frac{\partial}{\partial b} \end{aligned} \quad (35)$$

Left and right-invariant Haar measures are also easily obtained:

$$d^L g = \frac{1}{a} da \wedge db \quad (36)$$

$$d^R g = \frac{1}{a^2} da \wedge db \quad (37)$$

Since they are different, this group is not unimodular. Therefore we should take care of choosing the proper measure in each case.

Selecting the appropriate polarization, different (equivalent in this case) representations can be obtained for the affine group.

4.1 Representation on $L^2(\mathbb{R})$

Let us consider the polarization subalgebra $\mathcal{P}_{\mathbb{R}} = \langle X_a^R \rangle$ (in this case the right polarization is the most convenient). Therefore, the representation space is the subspace of $\mathcal{F}(G)$ of functions $\Psi(g)$ satisfying $X_a^R \Psi = \alpha \Psi$, $\alpha \in \mathbb{C}$. The solutions have the form:

$$\Psi(a, b) = W_{\alpha}(a, b) \phi(b) = a^{\alpha} \phi(b), \quad \phi \in L^2(\mathbb{R}, d\mu) \quad (38)$$

Later we shall determine the values of α and the measure $d\mu$ in order to have a unitary representation.

The representation is realized by the operators X_a^L and X_b^L acting on this space of functions. Since the factor a^{α} is common to all functions, it is convenient to subtract it and consider the realization in $L^2(\mathbb{R}, d\mu)$. Therefore, as in eq. (23), we define the operators:

$$\begin{aligned} \mathcal{X}_a &\equiv a^{-\alpha} (iX_a^L) a^{\alpha} = ib \frac{\partial}{\partial b} + i\alpha \\ \mathcal{X}_b &\equiv a^{-\alpha} (iX_b^L) a^{\alpha} = i \frac{\partial}{\partial b} \end{aligned} \quad (39)$$

Following the general construction, the measure $d\mu$ can be computed as $d\mu = i_{X_a^R} d^R g = \frac{1}{a} db$. This is not a well-defined measure on \mathbb{R} , and this reflects the fact that the quotient space $G/A \approx \mathbb{R}$ does not admit an invariant measure. Therefore we should look for quasi-invariant measures and introduce the appropriate Radon-Nikodym derivative. In this simple case, this amounts to restrict the allowed values of α to $\alpha = \frac{1}{2} + is$, $s \in \mathbb{R}$. In this way, if $\Psi = a^{\frac{1}{2}+is} \phi(b)$ and $\Psi' = a^{\frac{1}{2}+is} \phi'(b)$ are two functions, the scalar product is:

$$\langle \Psi, \Psi' \rangle = \int_{-\infty}^{\infty} \frac{db}{a} a^{\frac{1}{2}-is} \bar{\phi}(b) a^{\frac{1}{2}+is} \phi'(b) = \int_{-\infty}^{\infty} db \bar{\phi}(b) \phi'(b) \quad (40)$$

from which we deduce that the representation is unitary with the measure $d\mu = db$. The Radon-Nikodym derivative appears automatically as the factor $a^{\frac{1}{2}}$ in the functions Ψ . We shall leave for the next subsection the discussion of the irreducibility.

The finite group action for this representation is given by the right action. When restricted to $L^2(\mathbb{R})$ is:

$$\mathcal{U}_{\alpha}(g) \equiv a^{-\alpha} \mathcal{U}^R(g^{-1}) a^{\alpha} \quad (41)$$

in such a way that

$$\mathcal{U}_{\alpha}(a', b') \phi(b) = (a')^{-\alpha} \phi\left(\frac{b-b'}{a'}\right) \quad (42)$$

The representations \mathcal{U}_α with $\alpha = \frac{1}{2} + is$, $s \neq 0$ are unitarily equivalent to the representation $\mathcal{U}_{\frac{1}{2}}$, with $s = 0$, through the intertwining operator $T_s = a^{is}$, $\mathcal{U}_{\frac{1}{2}+is}T_s = T_s\mathcal{U}_{\frac{1}{2}}$. Therefore, we shall always work with $\alpha = \frac{1}{2}$, and denote $\mathcal{U} \equiv \mathcal{U}_{\frac{1}{2}}$.

The machinery of wavelets analysis proceeds now defining, for an admissible function $\gamma \in L^2(\mathbb{R})$:

$$\gamma_{b',a'}(b) = \mathcal{U}(a', b')\gamma(b) = (a')^{-\frac{1}{2}}\gamma\left(\frac{b-b'}{a'}\right) \quad (43)$$

4.2 Representation on $L^2(\mathbb{R}^+, \frac{da}{a})$

Let us consider now the polarization $\mathcal{P}_{\mathbb{R}^+} = \langle X_b^L \rangle$ (the left polarization is the best choice now). Therefore, the representation space is the subspace of $\mathcal{F}(G)$ of functions $\Psi(g)$ satisfying $X_b^L \Psi = \beta \Psi$, $\beta \in \mathbb{C}$. The solutions have the form:

$$\Psi(a, b) = W_\beta(a, b)\phi(a) = e^{\beta b}\phi(a), \quad \phi \in L^2(\mathbb{R}^+, d\mu) \quad (44)$$

As before, we shall determine later the values of the complex parameter β and the measure $d\mu$ in order to have a unitary representation.

The representation is realized, in this case, by the operators X_a^R and X_b^R acting on this space of functions. Since the factor $e^{\beta b}$ is common to all functions, it is convenient to subtract it and consider the realization in $L^2(\mathbb{R}^+, d\mu)$. Therefore we define the operators:

$$\begin{aligned} \mathcal{X}_a &\equiv e^{-\beta b}(iX_a^R)e^{\beta b} = ia\frac{\partial}{\partial a} \\ \mathcal{X}_b &\equiv e^{-\beta b}(iX_b^R)e^{\beta b} = i\beta a \end{aligned} \quad (45)$$

Following the general construction, the measure $d\mu$ can be computed as $d\mu = i_{X_b^L} d^L g = \frac{da}{a}$. In this case, we obtain a well defined measure on \mathbb{R}^+ and no Radon-Nikodym derivative is needed. The representation is unitary if $\rho \equiv i\beta \in \mathbb{R}$. If $\beta \neq 0$, we can define $r \equiv \rho a$ in such a way that

$$\begin{aligned} \mathcal{X}_a &= ir\frac{\partial}{\partial r} \\ \mathcal{X}_b &= r \end{aligned} \quad (46)$$

and $d\mu = \frac{dr}{|r|}$. This representation is unitary and irreducible on $L^2(\mathbb{R}^+, \frac{dr}{r})$ if $\rho > 0$ and on $L^2(\mathbb{R}^-, -\frac{dr}{r})$ if $\rho < 0$.

If $\beta = 0$, it turns out that $\mathcal{X}_b = 0$, and the representation is no longer irreducible. It rather decomposes in the direct integral of the eigenspaces of the operator \mathcal{X}_a .

The case $\beta = 0$ will be discarded, and the nonequivalent cases are given by $\rho = \pm 1$. We shall focus on the case $\rho = 1$, the other can be dealt in an analogous way. The finite group action is given by the left action, which, when restricted to $L^2(\mathbb{R}^+, \frac{da}{a})$, is given by:

$$\mathcal{U}(g) \equiv e^{ib}\mathcal{U}^L(g)e^{-ib} \quad (47)$$

in such a way that

$$\mathcal{U}(a', b')\phi(a) = e^{-iab'}\phi(a'a) \quad (48)$$

We wonder now what is the relation between the two unitary irreducible representations here obtained on \mathbb{R}^+ and \mathbb{R}^- , and the unitary representation on \mathbb{R} obtained in the previous subsection. The answer is provided by the Fourier transform, which tells us that $L^2(\mathbb{R}, db)$ is

reducible under the action of the affine group and decomposes as $L^2(\mathbb{R}) = H_+^2(\mathbb{R}) \oplus H_-^2(\mathbb{R})$, where $H_+^2(\mathbb{R})$ and $H_-^2(\mathbb{R})$ are the subspaces of progressive and regressive wavelets, respectively [24]. Therefore, the Fourier transform is a unitary isomorphism between $H_+^2(\mathbb{R})$ and $L^2(\mathbb{R}^+, \frac{da}{a})$ and between $H_-^2(\mathbb{R})$ and $L^2(\mathbb{R}^-, -\frac{da}{a})$.

Let us construct, in an explicit way, this unitary isomorphism. This can be achieved by means of the “polarization-changing operators” introduced in Sec. 3.3. Using Dirac’s notation, we can write formally:

$$\tilde{\phi}(b) = \langle b|\phi \rangle = \int_0^\infty \frac{da}{a} \langle b|a \rangle \langle a|\phi \rangle = \int_0^\infty \frac{da}{a} \langle b|a \rangle \phi(a) \quad (49)$$

where the completeness relation $I = \int_0^\infty \frac{da}{a} |a\rangle \langle a|$ has been introduced. The factor $\langle b|a \rangle$ in the integral is the kernel for the integral transform (49) which relates the representation on $L^2(\mathbb{R})$ with the one on $L^2(\mathbb{R}^+, \frac{da}{a})$. To compute this integral kernel we can proceed as explained in Sec. 3.3. For the basis $\{|n\rangle\}$ we can use the eigenfunctions of the “translation” operator \mathcal{X}_b , $\{|\beta\rangle\}$, $\beta \in \mathbb{R}$, which in $L^2(\mathbb{R})$ are $\langle b|\beta \rangle = e^{i\beta b}$, whereas in $L^2(\mathbb{R}^+, \frac{da}{a})$ write $\langle a|\beta \rangle = \delta(a - \beta)$ (in this case we must consider more general Hilbert spaces to account for distributions). Note that $\langle a|\beta \rangle$ is identically zero for $\beta < 0$ (something similar happens in $L^2(\mathbb{R}^-, -\frac{da}{a})$ for $\beta > 0$), and this is the clue for the reducibility of the representation of the affine group on $L^2(\mathbb{R})$. Therefore, we can write:

$$\langle b|a \rangle = \int_{-\infty}^\infty d\beta \langle b|\beta \rangle \langle \beta|a \rangle = \int_0^\infty d\beta e^{i\beta b} \delta(a - \beta) = e^{iab} \quad (50)$$

Thus, equation (49) can be written as:

$$\tilde{\phi}(b) = \int_0^\infty \frac{da}{a} e^{iab} \phi(a) \quad (51)$$

Equation (51) would be the analog of the inverse Fourier transform between $L^2(\mathbb{R}^+, \frac{da}{a})$ and $L^2(\mathbb{R})$. More precisely, this is the composition of the ordinary inverse Fourier transform between $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R})$ (in fact $H_+^2(\mathbb{R})$) and the unitary isomorphism between $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^+, \frac{da}{a})$.

The direct Fourier transform is given by the adjoint operator, which has as integral kernel the conjugate $\overline{\langle b|a \rangle} = \langle a|b \rangle = e^{-iab}$.

In an analogous manner, for $a < 0$ we have the corresponding integral transform between $L^2(\mathbb{R})$ (or rather $H_-^2(\mathbb{R})$) and $L^2(\mathbb{R}^-, -\frac{da}{a})$.

4.3 Representation on the right half plane

We can still obtain another representation of the affine group, this time on the right half complex plane $\mathbb{C}^+ \equiv \{w \in \mathbb{C} / \Re(w) > 0\}$. For this purpose the complex polarization $\mathcal{P}_{\mathbb{C}^+} = \langle X_a^R + iX_b^R \rangle$ can be used.

To obtain the solutions of this polarization it is better to exploit the complex structure of the manifold of the affine group, and to perform the change of variable:

$$\begin{aligned} w &= a + ib \\ \bar{w} &= a - ib \end{aligned} \quad (52)$$

In the new variables, the solutions to the polarization $(X_a^R + iX_b^R)\Psi = \nu\Psi$, $\nu \in \mathbb{C}$ are of the form:

$$\Psi(w, \bar{w}) = W_\nu(w, \bar{w})\phi(w) = (\Re(w))^{\frac{\nu}{2}}\phi(w) \quad (53)$$

That is, the carrier space of the representation is, apart from a common factor $(\Re(w))^{\frac{\nu}{2}}$, the space of holomorphic functions over \mathbb{C}^+ .

The operators realizing the representation are left invariant vector fields acting on this space of functions. If we subtract the common dependence in $(\Re(w))^{\frac{\nu}{2}}$, the resulting operators are:

$$\begin{aligned} \mathcal{X}_a &\equiv (\Re(w))^{-\frac{\nu}{2}}(iX_a^L)(\Re(w))^{\frac{\nu}{2}} = iw\frac{\partial}{\partial w} + i\frac{\nu}{2} \\ \mathcal{X}_b &\equiv (\Re(w))^{-\frac{\nu}{2}}(iX_b^L)(\Re(w))^{\frac{\nu}{2}} = -\frac{\partial}{\partial w} \end{aligned} \quad (54)$$

The measure in this case is the very right-invariant Haar measure, written in these coordinates, $d\mu = (\Re(w))^{-2}dw \wedge d\bar{w}$. The representation obtained is irreducible and unitary for $\nu \in \mathbb{R}$.

The finite group action, restricted to the holomorphic part of the functions, is given by the right action:

$$\mathcal{U}_\nu \equiv (\Re(w))^{-\frac{\nu}{2}}\mathcal{U}^R(\Re(w))^{\frac{\nu}{2}} \quad (55)$$

having the form:

$$\mathcal{U}_\nu(a', b')\phi(w) = (a')^{-\frac{\nu}{2}}\phi\left(\frac{w - ib'}{a'}\right) \quad (56)$$

All these representations for $\nu \in \mathbb{R}$ are equivalent to the case $\nu = 0$.

The polarization-changing operators that connect this representation with others previously obtained could also be computed. It is particularly interesting the one relating this representation to the one on $L^2(\mathbb{R}^+, \frac{da}{a})$, which is the Laplace transform. However, we shall postpone its explicit computation until we study the representations of the $SL(2, \mathbb{R})$ group, since there, with the help of the generating function for the generalized Laguerre polynomial, the computation will be straightforward (see Sec. 5.1).

5 CWT on \mathbb{R} and \mathbb{S}^1 from $G = SL(2, \mathbb{R})$

In this Section we shall identify the group of affine transformations of \mathbb{S}^1 and \mathbb{R} , namely $SL(2, \mathbb{R})$, and we shall derive the CWT on both spaces in a unified manner, based on the construction of general CS associated to square integrable representations (modulo a subgroup) of $SL(2, \mathbb{R})$ obtained from different (first- or higher-order) polarization subalgebras. We shall see that both transforms coincide in the Euclidean limit.

Our main ingredient will be the composition group law $g'' = g'g$ for $SL(2, \mathbb{R})$. In order to compute it, let us use the Iwasawa decomposition (see, for instance, [15]) to parameterize the $SL(2, \mathbb{R})$ group:

$$g = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \sqrt{a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\cos \theta}{\sqrt{a}} & \frac{b \cos \theta}{\sqrt{a}} - \sqrt{a} \sin \theta \\ \frac{\sin \theta}{\sqrt{a}} & \sqrt{a} \cos \theta + \frac{b \sin \theta}{\sqrt{a}} \end{pmatrix} \quad (57)$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$.

Using matrix multiplication we can derive the composition law for the parameters $\{a, b, \theta\}$, in the form $g'' = g'g$, with:

$$\begin{aligned} a'' &= \frac{a a'}{\cos^2 \theta + (a'^2 + b'^2) \sin^2 \theta + b' \sin 2\theta} \\ b'' &= \frac{(b + a b') \cos^2 \theta + \left(2 b b' + a \left(-1 + a'^2 + b'^2\right)\right) \cos \theta \sin \theta + \left(a'^2 b + b' (-a + b b')\right) \sin^2 \theta}{\cos^2 \theta + (a'^2 + b'^2) \sin^2 \theta + b' \sin 2\theta} \\ \tan \theta'' &= \frac{a' \cos \theta' \sin \theta + (\cos \theta + b' \sin \theta) \sin \theta'}{\cos \theta \cos \theta' + \sin \theta (b' \cos \theta' - a' \sin \theta')} \end{aligned} \quad (58)$$

Left and right-invariant vector fields can be easily derived from the group law:

$$\begin{aligned} X_a^L &= a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} \\ X_b^L &= \frac{\partial}{\partial b} \\ X_\theta^L &= -2 a b \frac{\partial}{\partial a} + (a^2 - b^2 - 1) \frac{\partial}{\partial b} + a \frac{\partial}{\partial \theta} \end{aligned} \quad (59)$$

$$\begin{aligned} X_a^R &= a \cos 2\theta \frac{\partial}{\partial a} + a \sin 2\theta \frac{\partial}{\partial b} + \cos \theta \sin \theta \frac{\partial}{\partial \theta} \\ X_b^R &= -a \sin 2\theta \frac{\partial}{\partial a} + a \cos 2\theta \frac{\partial}{\partial b} - \sin^2 \theta \frac{\partial}{\partial \theta} \\ X_\theta^R &= \frac{\partial}{\partial \theta} \end{aligned} \quad (60)$$

The commutation relations for left-invariant vector fields (for right-invariant ones they are the same except for the sign) is:

$$\begin{aligned} [X_a^L, X_b^L] &= -X_b^L \\ [X_a^L, X_\theta^L] &= 2X_b^L + X_\theta^L \\ [X_b^L, X_\theta^L] &= -2X_a^L \end{aligned} \quad (61)$$

The group $SL(2, \mathbb{R})$, being a simple group of rank one, has only one quadratic Casimir [15]. It can be checked by direct computation that the following quadratic operator commutes with all left and right invariant vector fields, and therefore is the Casimir operator:

$$\begin{aligned} \hat{C} &= (X_a^L)^2 + (X_b^L)^2 + \frac{1}{2} (X_b^L X_\theta^L + X_\theta^L X_b^L) \\ &= a^2 \left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} \right) + a \frac{\partial^2}{\partial b \partial \theta} \end{aligned} \quad (62)$$

It will serve us to complete higher-order polarization subalgebras.

As $SL(2, \mathbb{R})$ is unimodular, left- and right-invariant integration measures coincide. They have the form:

$$d^L g = \frac{da}{a^2} db d\theta = d^R g. \quad (63)$$

5.1 Affine wavelets on \mathbb{R}^+ from discrete series of $SL(2, \mathbb{R})$

We are seeking after $X = \mathbb{R}^+$ as the carrier space of the representation of G ; this implies that X_b^L and X_θ^L should be inside our polarization subalgebra, but not X_a^L . Unfortunately, looking at the third commutator in (61), it prevents us from a proper first-order polarization. However, it is always possible to complete X_b^L with the Casimir operator (62) and to consider the second-order polarization subalgebra $\mathcal{P}_{\mathbb{R}^+} = \langle \hat{C}, X_b^L \rangle$. The solution to the polarization equations:

$$\begin{cases} \hat{C}\Psi = q\Psi \\ X_b^L\Psi = \beta\Psi \end{cases} \quad (64)$$

has the form:

$$\Psi(a, b, \theta) = e^{\beta b} \psi(a, \theta) \quad (65)$$

where $\psi(a, \theta)$ satisfies the second order parabolic partial differential equation:

$$\left[a^2 \frac{\partial^2}{\partial a^2} + a\beta \frac{\partial}{\partial \theta} + (\beta^2 a^2 - q) \right] \psi(a, \theta) = 0 \quad (66)$$

This is a parabolic partial differential equation similar to the Heat equation, which can be solved working out the value of $\frac{\partial}{\partial \theta} \psi$, thus we can formally write the solutions to this second-order partial differential equation as:

$$\psi(a, \theta) = e^{\theta \hat{\Theta}(a)} \varphi(a) \quad (67)$$

where

$$\hat{\Theta}(a) \equiv -\frac{1}{\beta a} \left[a^2 \frac{\partial^2}{\partial a^2} + (\beta^2 a^2 - q) \right] \quad (68)$$

is a second-order differential operator and φ is an arbitrary function of $a \in \mathbb{R}^+$. The action of the operators \mathcal{X} (from right-invariant vector fields X^R) on solutions (65), when restricted to functions φ , turns out to be (after some algebra):

$$\begin{aligned} \mathcal{X}_a \varphi(a) &\equiv e^{-\beta b - \theta \hat{\Theta}} i X_a^R e^{\beta b + \theta \hat{\Theta}} \varphi(a) = i a \varphi'(a) \\ \mathcal{X}_b \varphi(a) &\equiv e^{-\beta b - \theta \hat{\Theta}} i X_b^R e^{\beta b + \theta \hat{\Theta}} \varphi(a) = i \beta a \varphi(a) \\ \mathcal{X}_\theta \varphi(a) &\equiv e^{-\beta b - \theta \hat{\Theta}} i X_\theta^R e^{\beta b + \theta \hat{\Theta}} \varphi(a) = i \hat{\Theta}(a) \varphi(a) \end{aligned} \quad (69)$$

As far as $\beta \neq 0$, the parameter β does not play any role, and can be eliminated defining $r = 2\rho a$, with $\rho = i\beta$. With this change of variable, our operators acquire the form:

$$\mathcal{X}_a = ir \frac{d}{dr}, \quad \mathcal{X}_b = \frac{r}{2}, \quad \mathcal{X}_\theta = 2r \frac{d^2}{dr^2} - \frac{r^2/2 + 2q}{r}. \quad (70)$$

One can easily verify that the operators (70) define a representation of the Lie algebra $sl(2, \mathbb{R})$. Moreover, the generators are hermitian if the measure $d\mu = dr/|r|$ is chosen and if $\rho \in \mathbb{R}$. For $\rho > 0$ the carrier space of the representation is \mathbb{R}^+ , but for $\rho < 0$ is \mathbb{R}^- . This representation is irreducible and it exponentiates to a unitary representation of the group $SL(2, \mathbb{R})$ if the Casimir eigenvalue verifies $q = k(k-1)$, with $k > 0$ half-integer (the Bargmann index). For $k > \frac{1}{2}$ this constitutes a representation of the discrete series of $SL(2, \mathbb{R})$, and therefore are square integrable representations.

For $\beta = 0$, the expressions obtained have no meaning. In this case, the solutions of Eq. (66) have a different form:

$$\psi(a, \theta) = a^\alpha \varphi(\theta) \quad (71)$$

where $\alpha(\alpha - 1) = q$. This case will be also recast later in Sec. 5.4 as a solution of a different polarization leading to the Continuous Series representation in the circumference. Thus, the second order polarization $\mathcal{P}_{\mathbb{R}^+}$ encompasses both series of representations.

We should note that this measure $d\mu = dr/|r|$ does not come directly from (63) as $da/a^2 = i_{X_b^L} i_{X_\theta^L} d^L g$ does. The reason is that $\mathcal{P}_{\mathbb{R}^+}$ is not a first-order, but a higher-order, polarization.

We can construct a canonical basis for the Hilbert space $L_q^2(\mathbb{R}^+, dr/r)$ through eigenfunctions of \mathcal{X}_θ as follows:

$$-\frac{1}{2}\mathcal{X}_\theta \varphi_n^k(r) = (k+n)\varphi_n^k(r) \Rightarrow \varphi_n^k(r) = e^{-r/2} r^k L_n^{2k-1}(r), \quad (72)$$

where $L_n^m(r)$ are generalized Laguerre polynomials fulfilling the standard equation:

$$r \frac{d^2 L_n^m}{dr^2} + (m+1-r) \frac{dL_n^m}{dr} + nL_n^m = 0. \quad (73)$$

Indeed, it can be easily proved that:

$$\langle \varphi_n^k | \varphi_m^k \rangle = \int_0^\infty e^{-r} r^{2k-1} L_n^{2k-1}(r) L_m^{2k-1}(r) dr = \frac{(n+2k-1)!}{n!} \delta_{nm}, \quad k \in \mathbb{N}/2. \quad (74)$$

Thus, the set

$$\mathcal{B}_k = \left\{ \langle r | kn \rangle = \phi_n^k(r) \equiv \frac{1}{\sqrt{N_n^k}} \varphi_n^k(r) \right\}, \quad N_n^k = \frac{(n+2k-1)!}{n!} \quad (75)$$

constitutes an orthonormal basis of $L_k^2(\mathbb{R}^+, dr/r)$.

Since the representations are square integrable (for half-integer $k > \frac{1}{2}$), coherent states can be defined on the whole group, leading to Klauder-Perelomov's coherent states (since these coherent states are invariant modulo a phase under the Cartan subgroup, they can be defined on the quotient $SL(2, \mathbb{R})/U(1)$).

However, since we are interested in affine wavelets in \mathbb{R} , or \mathbb{R}^+ in this case, we shall consider the restriction of the whole representation $U^L(g)$ of G to the affine subgroup, which consists of elements $g = \sigma(a, b) = (a, b, \theta = 0)$, and that can be obtained by exponentiation of the infinitesimal generators \mathcal{X}_a and \mathcal{X}_b (the factor $\frac{1}{2}$ appearing in \mathcal{X}_b in (70) comes from the definition $r = 2\rho a$, which has been introduced to obtain the standard Laguerre polynomials, and can be, of course, eliminated). Doing so, we recover the usual irreducible representation of the affine group on $L^2(\mathbb{R}^+, da/a)$, obtained in Sec. 4.2. Note that representations of G with different values of k are equivalent under the affine subgroup.

The construction of affine group wavelets $\gamma_{(a,b)}$ from coherent states $\gamma_{(a,b,\theta=0)}$ of G proceeds straightforwardly by taking sections $\sigma(a, b) = (a, b, \theta = 0)$.

5.2 Affine wavelets on the right half complex plane and on \mathbb{R}

For completeness, let us give the second-order polarization which provides signals on the right half complex plane. It is:

$$\mathcal{P}_{\mathbb{C}^+} = \langle \hat{C}, X_\theta^R \rangle. \quad (76)$$

We choose here right-(instead of left-)invariant vector fields to simplify expressions. The solution to the polarization conditions:

$$\begin{cases} \hat{C}\Psi = q\Psi \\ X_\theta^R\Psi = 2ik\Psi \end{cases} \quad (77)$$

(the factor $2i$ in the second equation has been introduced for convenience) has the form $\Psi(\theta, a, b) = e^{2ik\theta}\psi(a, b)$, where $\psi(a, b)$ satisfies the second order elliptic (Poisson-like) partial differential equation:

$$\left[a^2 \left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} \right) + 2ika \frac{\partial}{\partial b} - q \right] \psi(a, b) = 0. \quad (78)$$

The general solution to this elliptic equation can be obtained by a factorization method using complex variables. The solution is given in terms of arbitrary analytic functions φ of $w = a + ib$ as $\psi(w, \bar{w}) = (\Re(w))^k \varphi(w)$. The compatibility of the polarization conditions (77) reproduces the relation $q = k(k-1)$.

The operators realizing the representation are given now by left-invariant vector fields acting on wave functions defined on the whole $SL(2, \mathbb{R})$ group. This representation, when restricted to holomorphic functions $\varphi(w)$ on the half complex plane \mathbb{C}^+ , has the form:

$$\begin{aligned} \mathcal{X}_a \varphi(w) &\equiv a^{-k} e^{-2ik\theta} i X_a^L a^k e^{2ik\theta} \varphi(w) = ik \varphi(w) + iw \varphi'(w) \\ \mathcal{X}_b \varphi(w) &\equiv a^{-k} e^{-2ik\theta} i X_b^L a^k e^{2ik\theta} \varphi(w) = -\varphi'(w) \\ \mathcal{X}_\theta \varphi(w) &\equiv a^{-k} e^{-2ik\theta} i X_\theta^L a^k e^{2ik\theta} \varphi(w) = 2kw \varphi(w) - (w^2 - 1) \varphi'(w) \end{aligned} \quad (79)$$

Again, this is an irreducible representation of $sl(2, \mathbb{R})$, this time on functions ψ with support on \mathbb{C}^+ . The generators are also hermitian with the measure $d\mu = i_{X_\theta^L} d^L g = a^{-2} da db = \Re(w)^{-2} dw d\bar{w}$. The resulting Hilbert space $L_k^2(\mathbb{C}^+, d\mu)$, with scalar product

$$\langle \psi | \psi' \rangle = \int_{\mathbb{C}^+} \bar{\varphi}(w) \varphi'(w) (\Re(w))^{2(k-1)} d\bar{w} dw, \quad (80)$$

is isomorphic to the space $L_k^2(D, d\nu)$ of functions $\psi(z, \bar{z}) = (1 - |z|^2)^k \varphi(z)$, $[\varphi(z)$ holomorphic], on the open unit disk $D = \{z \in \mathbb{C}, |z| < 1\}$ with integration measure $d\nu = \frac{1}{(1-|z|^2)^2} d\bar{z} dz$. The transformation that maps the half plane onto the disk is $z = \frac{w-1}{w+1}$. An orthonormal basis of $L_k^2(\mathbb{C}^+, d\mu)$ is given by:

$$\begin{aligned} \mathcal{B}_k &= \left\{ \langle w | kn \rangle = \phi_n^k(w) \equiv \frac{1}{\sqrt{M_n^k}} \Re(w)^k (1+w)^{-2k} \left(\frac{w-1}{w+1} \right)^n \right\} \\ M_n^k &= \frac{\pi n! (2k-2)!}{2^{4k-2} (2k+n-1)!}. \end{aligned} \quad (81)$$

Denoting $\tilde{\varphi}(w) = \langle w | \varphi \rangle$ and $\varphi(r) = \langle r | \varphi \rangle$, and inserting the completeness relation $1 = \int_0^\infty |r\rangle \langle r| dr/r$ we obtain a relation

$$\tilde{\varphi}(w) = \int_0^\infty \frac{dr}{r} \langle w | r \rangle \varphi(r) \quad (82)$$

between functions in the half-plane representation, $\tilde{\varphi}(w) \in L_k^2(\mathbb{C}^+, d\mu)$, and functions in the “dilation” representation, $\varphi(r) \in L_k^2(\mathbb{R}^+, dr/r)$, with $\langle w|r \rangle$ the kernel or “polarization-changing operator” (see Sec. 3.3 and Sec. 4.2). An explicit expression of this kernel can be calculated by inserting in $\langle w|r \rangle$ the completeness relation for the eigenfunctions of \mathcal{X}_θ , $1 = \sum_{n=0}^{\infty} |kn\rangle \langle kn|$, giving:

$$\langle w|r \rangle = \sum_{n=0}^{\infty} \phi_n^k(w) \bar{\phi}_n^k(r) = \frac{1}{2\sqrt{\pi(2k-2)!}} \Re(w)^k r^k e^{-r\frac{w}{2}} \quad (83)$$

where we have made use of the generating function of the generalized Laguerre polynomials. Therefore, the “polarization-changing operator” turns out to be:

$$\tilde{\varphi}(w) = \frac{1}{2\sqrt{\pi(2k-2)!}} \Re(w)^k \int_0^{\infty} \frac{dr}{r} r^k e^{-r\frac{w}{2}} \varphi(r) \quad (84)$$

This integral transformation is unitary, and is nothing other than the Laplace transform. This is easily seen if we subtract the common weights $\Re(w)^k$ and $r^{-(k-1)}$ from the wave functions:

$$(\Re(w)^{-k} \tilde{\varphi}(w)) = [\mathcal{L}(r^{k-1} \varphi(r))](w) = \frac{1}{2\sqrt{\pi(2k-2)!}} \int_0^{\infty} dr e^{-r\frac{w}{2}} (r^{k-1} \varphi(r)) \quad (85)$$

In particular, the basis (75) and (81) are transformed into each other under \mathcal{L} .

The second order elliptic equation (78) involves the variables a and b , and we have solved it in terms of the complex variable $w = a + ib$. In principle, we could solve it in terms of the variable a , obtaining a realization in \mathbb{R}^+ equivalent to the one given in Sec. 5.1 (discrete series). We could also solve the equation in terms of b , obtaining a realization in \mathbb{R} which would provide the discrete series of representations with support in \mathbb{R} . These representations have a basis formed by the Relativistic Hermite Polynomials (since this representation can be related to a model of a relativistic harmonic oscillator), which are directly related to the Gegenbauer polynomials [21, 20].

In this way, restricting to the affine subgroup, we could recover the affine wavelets in \mathbb{R} given in Sec. 4.1. However, we shall not pursue in this direction since the computations are involved and the results have already been obtained in Sec. 4.1.

5.3 Polarization $< \hat{C}, X_a^L >$

For the sake of completeness, we shall consider the polarization $\mathcal{P}_{\mathbb{S}^1}^{HO} = < \hat{C}, X_a^L >$, although the solutions of this higher-order polarization are the same as the ones for the first-order polarization that will be given in Sec. 5.4. The polarization equations are:

$$\begin{cases} \hat{C}\Psi = q\Psi \\ X_a^L \Psi = \alpha\Psi \end{cases} \quad (86)$$

The solution of the second equation is $\Psi(a, b, \theta) = a^\alpha \phi(\tau, \theta)$, where $\tau \equiv \frac{b}{a}$. The first equation then leads to:

$$\left[(1 + \tau^2) \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \theta \partial \tau} + 2\tau(1 - \alpha) \frac{\partial}{\partial \tau} \right] \phi(\tau, \theta) = (q - \alpha(\alpha - 1)) \phi(\tau, \theta) \quad (87)$$

This is an hyperbolic equation which, for $q = \alpha(\alpha - 1)$ admits a very simple factorization:

$$\left[(1 + \tau^2) \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \theta} - 2(\alpha - 1)\tau \right] \left[\frac{\partial}{\partial \tau} \right] \phi(\tau, \theta) = 0 \quad (88)$$

Therefore its solutions are $\phi(\tau, \theta) = \phi(\theta)$. We shall not proceed further in the details of this polarization since the solutions are the same as for the first-order polarization in Sec. 5.4 (see below), even though we have in this case a second-order polarization. The reason lies in the factorization of the second-order partial differential equation into a product of two first-order differential operators. The solutions $\Psi(a, b, \theta) = a^\alpha \phi(\theta)$ really satisfy a couple of first-order differential equations.

A different factorization of the equation (87) is possible, but with $q = \alpha(\alpha + 1)$. In this case the representation obtained has support not in θ , but in the variable $\theta - \arctan \tau$. The representation is, however, equivalent to the previous one and to the representation obtained in Sec. 5.4, leading to the continuous series of representations with support on \mathbb{S}^1 .

5.4 Continuous Series in the circumference

Now we are searching for $X = \mathbb{S}^1$ as the carrier space of the representation of G ; this implies that X_b^L and X_a^L have to be inside our polarization subalgebra. Looking at the first commutator in (61), we see that both vector fields already close a first-order polarization subalgebra, which we shall denote by $\mathcal{P}_{\mathbb{S}^1} = \langle X_a^L, X_b^L \rangle$. The polarization conditions read:

$$\begin{cases} X_a^L \Psi = \alpha \Psi \\ X_b^L \Psi = \beta \Psi \end{cases} \quad (89)$$

Notice that, in particular, the commutation relations (61) force $\beta = 0$, that is, the character of the polarization subalgebra is trivial on the derived algebra, as explained in Sec. 3.1. The solution to these polarization equations has the form:

$$\psi^\alpha(a, b, \theta) = W_\alpha(a, b, \theta) \gamma(\theta) = a^\alpha \gamma(\theta). \quad (90)$$

Note that solutions ψ do not depend on b . The operators realizing the representation are given by right-invariant vector fields acting on wave functions defined on the whole $SL(2, \mathbb{R})$ group. Their restriction $\mathcal{X} = a^{-\alpha} i X^R a^\alpha$ to functions γ supported on the circumference turns out to be:

$$\mathcal{X}_a = \frac{i}{2} \sin 2\theta \frac{d}{d\theta} + i\alpha \cos 2\theta, \quad \mathcal{X}_b = \frac{i}{2} (\cos 2\theta - 1) \frac{d}{d\theta} - i\alpha \sin 2\theta, \quad \mathcal{X}_\theta = i \frac{d}{d\theta}. \quad (91)$$

Following the general construction, the measure $d\mu$ can be computed as $d\mu = i_{X_a^L} i_{X_b^L} d^L g = \frac{d\theta}{a}$, which is again an ill-defined measure on \mathbb{S}^1 . To obtain an appropriate measure with the corresponding Radon-Nikodym derivatives we must restrict to the values $\alpha = \frac{1}{2} + is, s \in \mathbb{R}$ (compare with the case of the affine group in Sec. 4.1), and the resulting scalar product is given by:

$$\langle \Psi, \Psi' \rangle = \int_{-\pi}^{\pi} \frac{d\theta}{a} a^{\frac{1}{2}-is} \bar{\gamma}(\theta) a^{\frac{1}{2}+is} \gamma'(\theta) = \int_{-\pi}^{\pi} d\theta \bar{\gamma}(\theta) \gamma'(\theta). \quad (92)$$

These representations are however not irreducible, and decompose into the direct sum of two irreducible representations, labelled by the representations of the cyclic subgroup $\mathbb{Z}_2 =$

$\{e, -e\}$ of $SL(2, \mathbb{R})$ (e denotes the identity element). This means that, in order to achieve the complete irreducibility, a non-connected polarization subgroup P is required. The representation associated with the trivial representation of \mathbb{Z}_2 leads to the Continuous Series of representations C_q^0 , with $q = 1/4 + s^2$, of $SL(2, \mathbb{R})$, which are also representations of $SO(2, 1) = SL(2, \mathbb{R})/\mathbb{Z}_2$ (see, for instance, [19] and [15]). If we consider the non-trivial representation of \mathbb{Z}_2 , the representations obtained are the Continuous Series $C_q^{1/2}$, which are representations of $SL(2, \mathbb{R})$ but not of $SO(2, 1)$ [§].

Note that, since \mathbb{Z}_2 is central in G , imposing that \mathbb{Z}_2 acts trivially in the finite right action [by the polarization equation (20)], also means that it acts trivially in the finite left action (13) and therefore, the transformation by $g = -e$ (a rotation by π) acts trivially in the space of solutions of the representations C_q^0 ; that is, a rotation by π keeps the functions unchanged, and therefore $\gamma(\theta)$ has periodicity π instead of 2π . This means that the true representation space is $L^2((-\frac{\pi}{2}, \frac{\pi}{2}), d\theta)$ instead of $L^2((-\pi, \pi), d\theta)$ (see the comments below on the definition of the dilation).

We shall restrict ourselves to the Continuous Series of representations C_q^0 , which are also representations of $SO(2, 1)$, and in particular to the representation $C_{1/4}^0$, corresponding to $s = 0$ and $\alpha = \frac{1}{2}$, denoting just the operator $\mathcal{U} \equiv \mathcal{U}_{1/2}$. This representation (as all the others in the Continuous Series) is not square integrable. Thus, we proceed by restricting ourselves to the homogeneous space $Q = G/N$, where N is the subgroup of upper triangular matrices with ones in the diagonal. Taking the Borel section $\sigma(a, \theta) = (a, b = 0, \theta)$, it is relatively easy to compute the restricted finite group action $\mathcal{U}(q)\gamma(\theta) \equiv a^{-\frac{1}{2}}U^L(\sigma(q))a^{\frac{1}{2}}\gamma(\theta)$ of elements $q = (a, \vartheta) \in G/N$ on functions γ on \mathbb{S}^1 . In fact, using the group law (58) to compute $(a, 0, \vartheta)^{-1}(1, 0, \theta)$, after some algebra we get:

$$\gamma_{\vartheta, a}(\theta) \equiv \mathcal{U}_\alpha(a, \vartheta)\gamma(\theta) = \lambda(1/a, \theta - \vartheta)^\alpha \gamma((\theta - \vartheta)_{1/a}), \quad (93)$$

where

$$\theta_a \equiv \arctan(a \tan(\theta)), \quad \lambda(a, \theta) \equiv \frac{a}{a^2 + (1 - a^2) \cos^2(\theta)}, \quad (94)$$

are the action $\theta \rightarrow \theta_a$ of dilations on elements $\theta \in \mathbb{S}^1$ (see figure 1 for a geometrical interpretation of this transformation) and λ is a multiplier (4) which coincides with the Radon-Nikodym derivative $\frac{d\theta}{d\theta_a}$, respectively. The action of G/N on \mathbb{S}^1 is defined modulo π , as it was commented before. In fact, the action of the affine subgroup $(a, b, 0)$ is defined modulo π , as can be seen from the group law (58), the representation (91), and Eqs. (93) and (94), since θ appears always in the form of $\sin(2\theta)$ and $\cos(2\theta)$. Therefore, we shall restrict ourselves in the sequel to this half circumference. Note the difference between our projection from the center of the (half-)circle and the case of the stereographic projection from the south-pole to define a dilation on the sphere in [3]. The reason for this difference is that we have started with the group $SL(2, \mathbb{R})$ instead of $SO(2, 1) = SL(2, \mathbb{R})/\mathbb{Z}_2$, as would correspond to obtaining wavelets on the circle in the paper [5], where wavelets on the $(n-1)$ -sphere were studied from the group $SO_0(n, 1)$ (note that the case of the circle is a singular case, and this is why it does not fit in with the general scheme of [5]). The purpose of using $SL(2, \mathbb{R})$ instead of $SO(2, 1)$ is to study, with the same group and with the same parametrization, wavelets on the circle and on the real line in a unified manner, since the affine group can be obtained from $SL(2, \mathbb{R})$ by simply putting $\theta = 0$ in the Iwasawa decomposition of $SL(2, \mathbb{R})$. The price we must pay for this choice is that a quotient by

[§]All these representations are irreducible, except for $C_{1/4}^{1/2}$, with $s = 0$, which decomposes itself into two irreducible representations [19, 15].

\mathbb{Z}_2 must be done. This is reflected in the fact that the minimal parabolic subgroup for $SO(2, 1)$ is connected meanwhile for $SL(2, \mathbb{R})$ is disconnected [15], containing a \mathbb{Z}_2 factor. In fact, our polarization subgroup coincides with the minimal parabolic subgroup in this case.

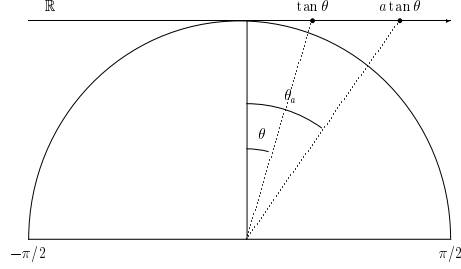


Figure 1: Action of a dilation $\theta \rightarrow \theta_a$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ lifted from the tangent line \mathbb{R} by inverse stereographic projection.

5.4.1 Admissibility condition

Let us investigate the general admissibility condition (9) for our particular case. Here

$$d^L q = i_{X_b^L} d^L g = \frac{da}{a^2} d\theta \quad (95)$$

is the measure on $Q = G/N$ projected from the measure (63) on the whole group. We shall prove that:

Theorem 5.1. *The representation \mathcal{U} given in (93) is square integrable mod (N, σ) and a non-zero vector $\gamma \in L^2((-\frac{\pi}{2}, \frac{\pi}{2}), d\theta)$ is admissible mod (N, σ) , iff there exists a constant $c > 0$ such that the quantity*

$$\Lambda_n = \int_0^\infty \frac{da}{a^2} |\hat{\gamma}_a^n|^2 < c, \quad \forall n \in \mathbb{Z}, \quad (96)$$

where $\hat{\gamma}_a^n = \langle n | \gamma_a \rangle$ represents the Fourier coefficient of $\gamma_a(\theta) = [\mathcal{U}(a, 0)\gamma](\theta)$ in the orthonormal basis $\langle \theta | n \rangle \equiv \frac{1}{\sqrt{\pi}} e^{2in\theta}$ of $L^2((-\frac{\pi}{2}, \frac{\pi}{2}), d\theta)$.

Proof. Let us rewrite the general condition (9) in our particular case. One can easily check that the transformation $\mathcal{U}(\theta, a)$ can be split into two parts $\mathcal{U}(a, \theta) = \mathcal{U}(1, \theta)\mathcal{U}(a, 0)$. The effect of $\mathcal{U}(1, \theta)$ on the Fourier coefficient of (93) is just a multiplicative phase factor, that is:

$$\hat{\gamma}_{\vartheta, a}^n = \langle n | \gamma_{\vartheta, a} \rangle = \frac{1}{\sqrt{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma_{\vartheta, a}(\theta) e^{-2in\theta} d\theta = e^{-2in\vartheta} \hat{\gamma}_a^n. \quad (97)$$

Using the Parseval's identity $1 = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} |n\rangle \langle n|$ for the Fourier series on $(-\frac{\pi}{2}, \frac{\pi}{2})$, the integrand of (9) can be written as:

$$|\langle \mathcal{U}(\vartheta, a)\gamma | \psi \rangle|^2 = \frac{1}{\pi} \sum_{m, n} e^{2i(n-m)\vartheta} \bar{\hat{\gamma}}_a^n \hat{\gamma}_a^m \hat{\psi}^n \bar{\hat{\psi}}^m, \quad (98)$$

and the admissibility condition (9) reads:

$$\int_0^\infty \frac{da}{a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta |\langle \gamma_{\vartheta,a} | \psi \rangle|^2 = \int_0^\infty \frac{da}{a^2} \sum_{n=-\infty}^\infty |\hat{\gamma}_a^n|^2 |\hat{\psi}^n|^2 = \sum_{n=-\infty}^\infty \Lambda_n |\hat{\psi}^n|^2, \quad (99)$$

where we have used orthogonality relations for trigonometric functions, and used the definition (96) of Λ_n . Taking into account that $\{|\hat{\psi}^n|^2\} \in \ell^1(\mathbb{Z})$, since $\sum_{n=-\infty}^\infty |\hat{\psi}^n|^2 = \|\psi\|^2$, the admissibility condition (9) adopts the following form:

$$\sum_{n=-\infty}^\infty |\hat{\psi}^n|^2 \Lambda_n < \infty, \quad \forall \{|\hat{\psi}^n|^2\} \in \ell^1(\mathbb{Z}), \quad (100)$$

which converges absolutely iff $\{\Lambda_n\} \in \ell^\infty(\mathbb{Z})$. That is, γ is admissible iff $\Lambda_n < c$, with c independent of n , which proves the theorem. ■

We shall postpone until the end of this Section 5.4 the proof of the existence of functions γ satisfying the condition (96).

The strong condition (96) is necessary and sufficient for the admissibility of γ , but it entails cumbersome evaluations of Fourier coefficients. We shall give a more maneuverable condition, although it will be only necessary.

Proposition 5.2. *A function $\gamma \in L^2((-\frac{\pi}{2}, \frac{\pi}{2}), d\theta)$ is admissible only if it fulfils the condition:*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\cos \theta} \gamma(\theta) = 0. \quad (101)$$

Proof. Let us assume that $\gamma(\theta) = 0$ if $|\theta| > \tilde{\theta} < \frac{\pi}{2}$ (i.e. the support of γ is bounded away from $\pm \frac{\pi}{2}$). Then, $\gamma(\theta_{1/a}) = 0$ if $|\theta_{1/a}| > \tilde{\theta} \Rightarrow |\theta| > \tilde{\theta}_a$, where θ_a defined in (94) is the transformed angle under dilations. Thus, the Fourier coefficient $\hat{\gamma}_a^n$ in (96) is calculated as:

$$\hat{\gamma}_a^n = \langle n | \gamma \rangle = \frac{1}{\sqrt{\pi}} \int_{-\tilde{\theta}_a}^{\tilde{\theta}_a} \lambda(1/a, \theta)^{1/2} \gamma(\theta_{1/a}) e^{-2in\theta} d\theta. \quad (102)$$

Performing the change of variable $\vartheta = \theta_{1/a}$ and taking into account that $\frac{d\theta_{1/a}}{d\theta} = \lambda(1/a, \theta)$ (the Radon-Nikodym derivative) and that $\lambda(1/a, \theta_a) = \lambda(a, \vartheta)^{-1}$, we have

$$\hat{\gamma}_a^n = \langle n | \gamma \rangle = \frac{1}{\sqrt{\pi}} \int_{-\tilde{\vartheta}}^{\tilde{\vartheta}} \lambda(a, \vartheta)^{1/2} \gamma(\vartheta) e^{-2in\vartheta_a} d\vartheta. \quad (103)$$

Let us split the integral (96) into three parts: short ($\epsilon \ll 1$), intermediate and large scales:

$$\int_0^\infty = \int_0^\epsilon + \int_\epsilon^{1/\epsilon} + \int_{1/\epsilon}^\infty. \quad (104)$$

For $a \ll 1$ we can approximate $\lambda(a, \vartheta)^{1/2} \approx \frac{\sqrt{a}}{\cos \vartheta}$. Therefore, the integral over small scales, $a < \epsilon \ll 1$

$$\int_0^\epsilon \frac{da}{a^2} |\hat{\gamma}_a^n|^2 \approx \frac{1}{\pi} \int_0^\epsilon \frac{da}{a} \left| \int_{-\tilde{\vartheta}}^{\tilde{\vartheta}} \frac{d\theta}{\cos \theta} \gamma(\theta) \right|^2, \quad (105)$$

converges only if the weak condition (101) holds (in fact, the result must be independent of $\tilde{\theta}$). At intermediate scales, the second integral in (104) is finite because the integrand $\hat{\gamma}_a^n$ is a bounded continuous function on $(\epsilon, 1/\epsilon)$; indeed, $\mathcal{U}(a, 0)$ is a strongly continuous operator and the scalar product is also continuous. It just remains to study the behavior at large scales $a \gg 1$. Performing the change of variable $a \rightarrow 1/a$ and using the behavior at short scales, we can write:

$$\hat{\gamma}_a^n \approx \frac{1}{\sqrt{\pi}} \int_{-\tilde{\theta}_a}^{\tilde{\theta}_a} \frac{\sqrt{1/a}}{\cos \theta} \gamma(\theta_{1/a}) e^{-2in\theta} d\theta, \quad a \gg 1. \quad (106)$$

The only large scale divergence in the third part of (104) will never be reached because $\gamma(\theta) = 0$ outside $(-\tilde{\theta}, \tilde{\theta})$ by assumption. Finally, if we drop the restriction on the support of γ , the condition (101) is just necessary, which proves the proposition. ■

The necessary condition (101) is the compact counterpart of the usual zero-mean condition $\int_{-\infty}^{\infty} dx \gamma(x) = 0$ for affine wavelets on \mathbb{R} . We shall show in Sect. 5.5 that they coincide in the Euclidean limit (large radius).

5.4.2 Continuous frame condition

Taking into account the result (96), the admissibility condition (9,99) can be written as:

$$\int_0^{\infty} \frac{da}{a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta |\langle \gamma_{\vartheta,a} | \psi \rangle|^2 \leq c \|\psi\|^2. \quad (107)$$

This means that the family $\{\gamma_{\vartheta,a}, (\vartheta, a) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^+\}$ is a continuous family of coherent states. Moreover:

Proposition 5.3. *For any even admissible vector γ , the family $\{\gamma_{\vartheta,a}, (\vartheta, a) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^+\}$ is a continuous frame; that is, there exist constants $0 < c_1 \leq c_2$ such that*

$$c_1 \|\psi\|^2 \leq \int_0^{\infty} \frac{da}{a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta |\langle \gamma_{\vartheta,a} | \psi \rangle|^2 \leq c_2 \|\psi\|^2, \quad \forall \psi \in L^2((-\frac{\pi}{2}, \frac{\pi}{2}), d\theta). \quad (108)$$

Before tackling the proof of this proposition, we introduce a previous lemma, following the strategy of Ref. [3].

Lemma 5.4. (1) *The correspondence $S : L^2((-\frac{\pi}{2}, \frac{\pi}{2}), d\theta) \rightarrow L^2(\mathbb{R}, dx)$ defined by:*

$$\Gamma(x) = [S\gamma](x) = \frac{1}{\sqrt{1+x^2}} \gamma(\arctan x) \quad (109)$$

is an isometry and an unitary map.

(2) *Let us denote $\tilde{D}_a = \mathcal{U}(a, 0) : \gamma(\theta) \rightarrow \lambda(1/a, \theta)^{1/2} \gamma(\theta_{1/a})$ the dilation (93) on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $D_a : f(x) \rightarrow \frac{1}{\sqrt{a}} f(x/a)$ the usual dilation on \mathbb{R} . Then the intertwining relation $S\tilde{D}_a = D_a S$ holds.*

Both statements can be proved by a direct computation. We are now in condition to prove the proposition 5.3.

Proof of Proposition 5.3 It remains only to prove the lower bound. We claim that the quantity defined in (96) is strictly positive $\Lambda_n > 0, \forall n \in \mathbb{Z}$. Assume on the contrary that

$\Lambda_m = 0$ for some m . This is possible only if $\hat{\gamma}_a^m = 0, \forall a \in \mathbb{R}^+$. Let us write this Fourier coefficient in a different form using the previous Lemma:

$$\hat{\gamma}_a^m = \langle m | \tilde{D}_a \gamma \rangle_{L^2(-\frac{\pi}{2}, \frac{\pi}{2})} = \langle Sm | S \tilde{D}_a \gamma \rangle_{L^2(\mathbb{R})} = 2\sqrt{a} \int_0^\infty \frac{dx}{x} f(x) g(a/x), \quad (110)$$

with

$$f(x) = \frac{1}{\sqrt{1+x^2}} \cos(2m \arctan x), \quad g(a/x) = \frac{1}{\sqrt{1+(a/x)^2}} \gamma(\operatorname{arccotan}(a/x)), \quad (111)$$

where we have used that γ is even. Notice that the integral (110) has the form of a convolution in $L^2(\mathbb{R}^+, dx/x)$. Looking at $\hat{\gamma}_a^m$ in this way, one knows that the convolution is zero for all $a \in \mathbb{R}^+$ only if f or g are identically zero, which implies $\gamma = 0$. Therefore $\Lambda_n > 0$ for every n . To see that Λ_n is also bounded from below and $\Lambda_n > c(\gamma)$ independent of n , we first show that the main contribution to the integral (96) comes from the region $a \sim 1/|m| \ll 1$ with $|m| > M \gg 1$ (M fixed). Indeed, the Fourier coefficient

$$\hat{\gamma}_a^m = \frac{1}{\sqrt{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma_a(\theta) e^{-2in\theta} d\theta = \frac{1}{\sqrt{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda(a, \theta)^{1/2} \gamma(\theta) e^{-2in\theta_a} d\theta \quad (112)$$

is nearly zero for $|m| > M \gg 1$ (M fixed) and $a \gg \frac{1}{|m|}$. Then, the main contribution to $\hat{\gamma}_a^m$ for $|m| > M \gg 1$ must come from the range $a \in (\frac{1}{|m|} - \frac{1}{k|m|}, \frac{1}{|m|} + \frac{1}{k|m|})$, with $k > 1$. Using the approximations $\lambda(a, \vartheta)^{1/2} \approx \frac{\sqrt{a}}{\cos \vartheta}$ and $\theta_a \approx a\theta$ for small a , we can write $\hat{\gamma}_a^m \approx \sqrt{a} \kappa(\gamma)$ with $\kappa(\gamma) = \frac{1}{\sqrt{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\gamma(\theta)}{\cos(\theta)} e^{-2i\theta} d\theta$ independent of m . Thus, for large m , the asymptotic behavior of

$$\Lambda_m = \int_0^\infty \frac{da}{a^2} |\hat{\gamma}_a^m|^2 \gtrsim |\kappa(\gamma)|^2 \int_{\frac{1}{|m|} - \frac{1}{k|m|}}^{\frac{1}{|m|} + \frac{1}{k|m|}} \frac{da}{a} = |k(\gamma)|^2 \ln\left(\frac{k+1}{k-1}\right) \quad (113)$$

(for some finite $k > 1$) gives a strictly positive quantity independent of m , which proves that Λ_m is bounded from below. Notice that, in case $\kappa(\gamma)$ happened to be zero for a particular vector γ , we could always take $a \sim |l/m|$, with $|l| \ll |m| \gg 1$, so that the new lower bound $\kappa_l(\gamma) \neq 0$ for some l , otherwise $\gamma = 0$. ■

We believe the requirement of γ to be even in Proposition 5.3 is too strong and perhaps it can be relaxed to a weaker condition as to have a non-zero even part.

As in Ref. [3], we shall conjecture here that, for general admissible vectors, the frame $\{\gamma_{\vartheta, a}, (\vartheta, a) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^+\}$ is not tight. To get a tight frame would require an equality in (108). Thus, we also believe that the operator A_σ in (10) has a spectrum in a nontrivial interval, which should contract to a point in the Euclidean limit (see next Section). Nevertheless, there must be particular vectors γ for which one can construct tight frames, but we shall not discuss them here.

Now we are in condition to resume the question of the existence of admissible circular vectors γ by resorting to the well known Euclidean case. First we note that:

Lemma 5.5. *The usual wavelet admissibility condition $C_\Gamma = \int_{-\infty}^\infty |\hat{\Gamma}(k)|^2 \frac{dk}{|k|} < \infty$ for the Fourier transform $\hat{\Gamma}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-ikx} \Gamma(x)$ of functions $\Gamma \in L^2(\mathbb{R})$ on the real line, is equivalent to the condition*

$$I_\Gamma = \int_{-\infty}^\infty db \int_0^\infty \frac{da}{a^2} |\langle \Gamma_{b,a} | f \rangle|_{L^2(\mathbb{R})}^2 < \infty \quad (114)$$

for any $f \neq 0$ in $L^2(\mathbb{R})$, where $\Gamma_{b,a}(x) = a^{-\frac{1}{2}}\Gamma(\frac{x-b}{a})$ was defined in Eq. (43).

Proof. See, for example, the reference [4] ■

Next we can see that:

Proposition 5.6. *Any admissible wavelet on $L^2(\mathbb{R})$ yields an admissible wavelet on $L^2(-\frac{\pi}{2}, \frac{\pi}{2})$ by inverse stereographic projection (109).*

Proof. We shall follow the strategy of Ref. [25]. Let us split the general unitary operator $\mathcal{U}(a, \vartheta)$ in (93) into translation and dilation as $\mathcal{U}(a, \vartheta) = \mathcal{U}(1, \vartheta)\mathcal{U}(a, 0)$ and simply denote $R_\vartheta = \mathcal{U}(1, \vartheta)$ and $\tilde{D}_a = \mathcal{U}(a, 0)$, as before. Then, the admissibility condition (107) of a wavelet $\gamma \in L^2(-\frac{\pi}{2}, \frac{\pi}{2})$ can be written as:

$$I_\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta \int_0^\infty \frac{da}{a^2} |\langle R_\vartheta \tilde{D}_a \gamma | \psi \rangle|_{L^2(-\frac{\pi}{2}, \frac{\pi}{2})}^2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta I_\gamma(\vartheta) < \infty, \quad \forall \psi \in L^2(-\frac{\pi}{2}, \frac{\pi}{2}). \quad (115)$$

Using that the stereographic projection operator (109) is an isometry and an unitary map, and that R_ϑ is unitary, we have that:

$$I_\gamma(\vartheta) \equiv \int_0^\infty \frac{da}{a^2} |\langle S \tilde{D}_a \gamma | S R_\vartheta^{-1} \psi \rangle|_{L^2(\mathbb{R})}^2 = \int_0^\infty \frac{da}{a^2} |\langle D_a S \gamma | S R_\vartheta^{-1} \psi \rangle|_{L^2(\mathbb{R})}^2, \quad (116)$$

where we have used the intertwining relation $S \tilde{D}_a = D_a S$ in the last equality. If $\Gamma(x) \equiv [S\gamma](x)$ is an admissible wavelet on $L^2(\mathbb{R})$, the integral $I_\gamma(\vartheta)$ converges since $[S R_\vartheta^{-1} \psi] \in L^2(\mathbb{R})$ for any $\psi \in L^2(-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, for any $\psi \in L^2(-\frac{\pi}{2}, \frac{\pi}{2})$, the integral $I_\gamma(\vartheta)$ is a continuous bounded function of $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, due to the compactness of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, I_γ also converges, which means that γ is an admissible “circular” wavelet. ■

Interesting cases of admissible vectors are the “difference-of-Gaussians” $\gamma_\alpha(\theta) = \gamma(\theta) - [\tilde{D}_\alpha \gamma](\theta)$, with $\gamma(\theta) = e^{-\tan^2 \theta}$, which one can see (at least numerically) that verifies the necessary (weak) condition (101).

Another natural question is the covariance of the CWT. Using the group law (58) it is easy to see that the CWT on the circle is covariant under motions on $(-\frac{\pi}{2}, \frac{\pi}{2})$, but not under dilations. Indeed, the composition of two elements of the form $(\theta, a, b = 0)$ gives an element with $b \neq 0$.

5.5 The Euclidean limit

One would expect the wavelet transform on the circle to behave locally (at short scales or large values of the radius) like the usual (flat) wavelet transform. Let us see that this is the case.

5.5.1 Contracting $SL(2, \mathbb{R})$ to $\mathbb{R}^2 \rtimes \mathbb{R}^+$

A contraction $\mathcal{G}' = \mathbb{R}^2 \rtimes \mathbb{R}^+$ of $\mathcal{G} = sl(2, \mathbb{R})$ along the (identity connected component or the) minimal parabolic subgroup $P_0 = AN = SIM(1) = N \rtimes \mathbb{R}^+$ can be constructed through the one-parameter family of invertible linear mappings $\pi_R : \mathbb{R}^3 \rightarrow \mathbb{R}^3, R \in [1, \infty)$ defined by:

$$\pi_R(X_a) = X_a, \quad \pi_R(X_b) = X_b, \quad \pi_R(X_\theta) = R^{-1}X_\theta, \quad (117)$$

such that the Lie bracket of \mathcal{G}' is:

$$[X, Y]' = \lim_{R \rightarrow \infty} \pi_R^{-1}[\pi_R X, \pi_R Y], \quad (118)$$

with $[\cdot, \cdot]$ the Lie bracket (61) of \mathcal{G} . The resulting \mathcal{G}' commutators are:

$$[X_a, X_b]' = X_b, \quad [X_a, X_\theta]' = -X_\theta, \quad [X_b, X_\theta]' = 0. \quad (119)$$

The contraction process is lifted to the corresponding Lie groups considering the mapping:

$$\begin{aligned} \Pi_R : \mathbb{R}^2 \rtimes \mathbb{R}^+ &\rightarrow SL(2, \mathbb{R}) \\ (x, y, a) &\mapsto \Pi_R(x, y, a) = (\theta = \arctan \frac{x}{R}, b = y, a). \end{aligned} \quad (120)$$

Next we have to restrict the contraction map (120) to the homogeneous (parameter) spaces $G/N = Q$ and $G'/N = SIM(1)$ (notice that the group structure of the parameter space and covariance are restored after contraction); that is, we have to quotient out the nilpotent subgroup N , which is preserved under contraction. To that aim, we introduce a section $\sigma' : G'/N \rightarrow G'$ defined by $\sigma'(x, a) = (x, 0, a)$, which combined with the canonical projection of the Iwasawa bundle $p : G \rightarrow Q = G/N$, leads to the natural restricted contraction map:

$$\begin{aligned} \Pi_R^{\sigma'} : \mathbb{R} \rtimes \mathbb{R}_*^+ &\rightarrow SL(2, \mathbb{R})/N \\ q' = (b, a) &\mapsto q = \Pi_R^{\sigma'}(q') = p(\Pi_R(\sigma'(q'))) = (\arctan \frac{b}{R}, a). \end{aligned} \quad (121)$$

5.5.2 The Euclidean limit of the CWT on the circle

The Euclidean limit is formulated as a contraction at the level of group representations. We shall see (the procedure parallels that of Ref. [3]) that the continuous series representation of G contract to the usual wavelet representation of the affine group $SIM(1)$ in the following sense:

Definition 5.7. *Let G' be a contraction of G , defined by the contraction map $\Pi_R : G' \rightarrow G$, and let \mathcal{U}' be a representation of G' in a Hilbert space \mathcal{H}' . Let $\{\mathcal{U}_R\}$, $R \in [1, \infty)$ be a one-parameter family of representations of G on a Hilbert space \mathcal{H}_R , and $I_R : \mathcal{H}_R \rightarrow \mathcal{D}_R$ a linear injective map from \mathcal{H}_R onto a dense subspace $\mathcal{D}_R \subset \mathcal{H}'$. Then we shall say that \mathcal{U}' is a contraction of the family $\{\mathcal{U}_R\}$ if there exists a dense subspace $\mathcal{D}' \subset \mathcal{H}'$ such that, for all $f \in \mathcal{D}'$ and $g' \in G'$, one has:*

- For every R large enough, $f \in \mathcal{D}_R$ and $\mathcal{U}_R(\Pi_R(g'))I_R^{-1}f \in I_R^{-1}\mathcal{D}_R$.
- $\lim_{R \rightarrow \infty} \|I_R \mathcal{U}_R(\Pi_R(g'))I_R^{-1}f - \mathcal{U}'(g')f\|_{\mathcal{H}'} = 0, \quad \forall g' \in G'$.

In our case, $\mathcal{H}_R = L^2[(-\frac{\pi}{2}R, \frac{\pi}{2}R), Rd\theta] \approx \mathcal{H} = L^2[(-\frac{\pi}{2}, \frac{\pi}{2}), d\theta]$ is the Hilbert space of square integrable functions on the half-circumference of radius R and $\mathcal{H}' = L^2(\mathbb{R}, dx)$. It is easy to check that

$$\begin{aligned} I_R : \mathcal{H}_R &\rightarrow \mathcal{H}' \\ \gamma &\rightarrow [I_R \gamma](x) = \frac{1}{\sqrt{1 + (x/R)^2}} \gamma(\arctan(x/R)) \end{aligned} \quad (122)$$

is an isometry, i.e.: $\|I_R \gamma\|_{\mathcal{H}'} = \|\gamma\|_{\mathcal{H}_R}$. The inverse map

$$[I_R^{-1}f](\theta) = \frac{1}{\cos \theta} f(R \tan \theta)$$

is also an isometry. For each R , we choose $\mathcal{D}_R = \mathcal{D}' = C_0(\mathbb{R})$, the space of continuous functions of compact support. Then, we have to prove that:

Theorem 5.8. *The representation $[\mathcal{U}'(b, a)f](x) = \frac{1}{\sqrt{a}}f(\frac{x-b}{a})$ of the affine group $SIM(1)$ is a contraction of the one-parameter family \mathcal{U}_R of representations of $SL(2, \mathbb{R})$ on \mathcal{H}_R , given by (93) realized on \mathcal{H}_R , as $R \rightarrow \infty$. That is:*

$$\lim_{R \rightarrow \infty} \|I_R \mathcal{U}_R(\Pi_R^{\sigma'}(b, a))I_R^{-1}f - \mathcal{U}'(b, a)f\|_{\mathcal{H}'} = 0, \quad \forall (b, a) \in \mathbb{R} \rtimes \mathbb{R}^+ \quad (123)$$

Proof. By direct computation

$$\lim_{R \rightarrow \infty} |I_R \mathcal{U}_R(\Pi_R^{\sigma'}(b, a))I_R^{-1}f - \mathcal{U}'(b, a)f| = 0, \quad \forall (b, a) \in \mathbb{R} \rtimes \mathbb{R}^+ \quad (124)$$

which means pointwise convergence.

Now consider $f \in \mathcal{D}'$. Since f is continuous and of compact support, it is bounded. Since I_R , \mathcal{U}' and \mathcal{U}_R are unitary, the expression

$$|I_R \mathcal{U}_R(\Pi_R^{\sigma'}(b, a))I_R^{-1}f - \mathcal{U}'(b, a)f| < |I_R \mathcal{U}_R(\Pi_R^{\sigma'}(b, a))I_R^{-1}f| + |\mathcal{U}'(b, a)f| \quad (125)$$

is uniformly bounded and the left hand side converges pointwise to zero as $R \rightarrow \infty$. Then Lebesgue's dominated convergence theorem states that it tends to zero in the strong sense. ■

Conclusions and outlook

We have exposed a general approach to obtain the CWT on a manifold X associated to a (first or higher-order) polarization subalgebra \mathcal{P}_X of a symmetry group G of “affine transformations”. As an application, we have derived the CWT on \mathbb{R} and \mathbb{S}^1 entirely from the group $SL(2, \mathbb{R})$. Theorem 5.1 yields the basic ingredient for writing a genuine CWT (11) on \mathbb{S}^1 . We have provided admissibility conditions and proved that, for an admissible even vector γ , the family $\{\gamma_{\vartheta, a}\}$ is a continuous frame. The proposed CWT on the circle has the expected Euclidean limit; that is, it behaves locally like the usual (flat) CWT. We have given a precise mathematical meaning to these notions using the technique of group contractions.

In this article we mainly focused on “the continuous approach”, based on the theory of coherent states of quantum physics (formulated in terms of group representation theory). It remains to study “the discrete approach”, which roots in the Littlewood-Paley analysis, and yields fast algorithms for computing the wavelet transform numerically. An intermediate approach which paves the way between the continuous and the discrete cases is based on the representations of some finite groups [26, 27, 28]. We believe a thorough group-theoretic treatment of (finite) periodic sampled signals is possible, where the classification of filters amounts to a group-theoretic cohomology problem. However, the discussion of this interesting subject deserves a separate study.

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